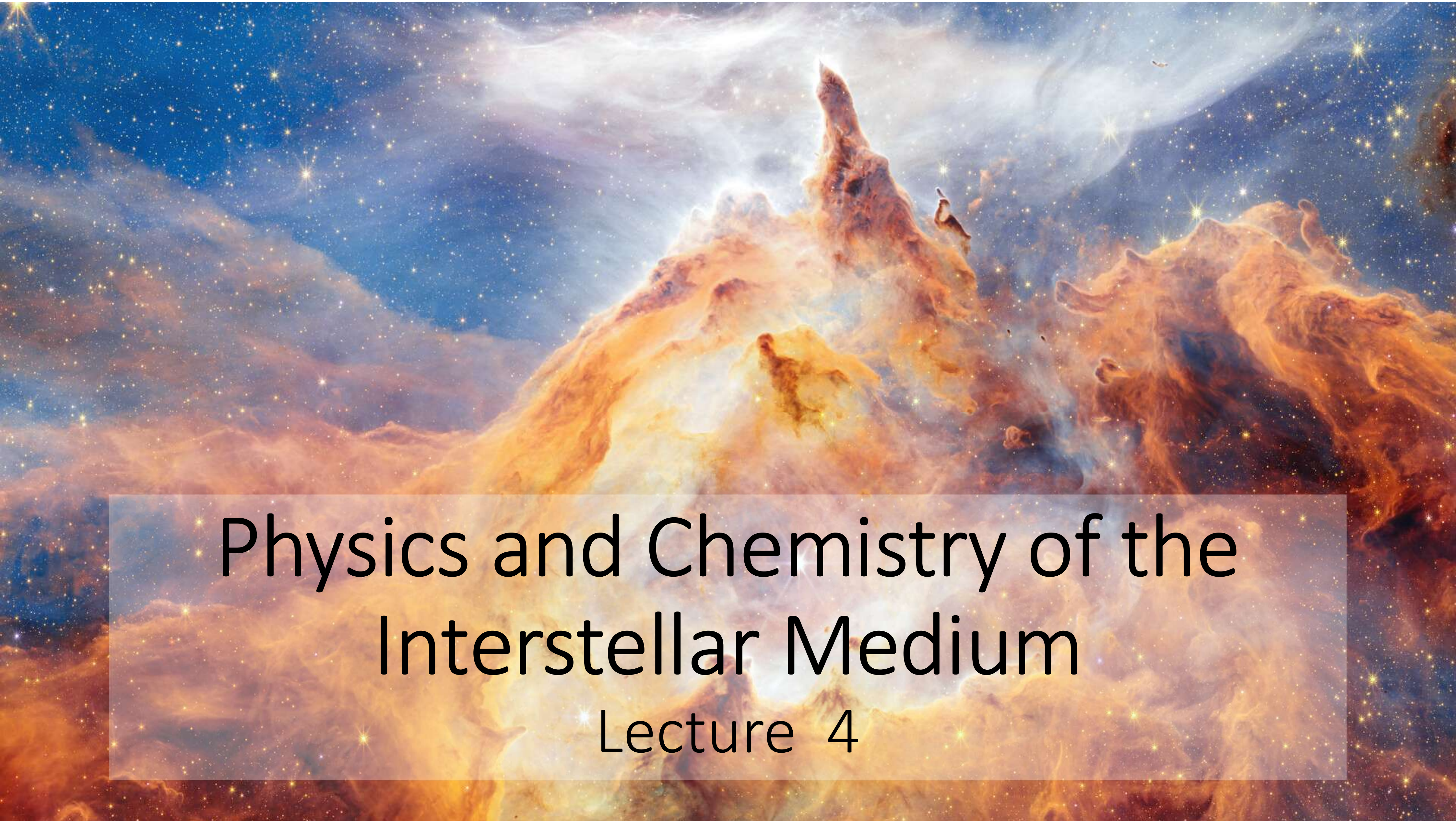


Physics and Chemistry of the Interstellar Medium

Lecture 4

The dynamics of interstellar gas

2. Instabilities and Turbulence

2.1 Jeans instability

2.2 Rayleigh-Taylor

2.3 Kelvin-Helmholtz

2.4 Parker instability

2.5 Turbulence

Kolmogorov scaling

Observations

Scaling relations

Gravitational instability – Jeans instability

- Pure hydrodynamics, i.e. neglect of $\vec{j}$ term
- Perturbation analysis for „smooth“ medium:
  - Assume
    - Extended (inifinite) medium
    - Constant density ρ_0
    - At rest $\vec{v}_0 = 0$
  - Evaluate
    - Evolution of a small density perturbation $\rho = \rho_0 + \rho_1, \quad \rho_1 \ll \rho_0$

      $\frac{d\rho}{dt} = \frac{d\rho_1}{dt} \qquad \vec{v} = \vec{v}_1$

    - Small perturbation does not lead to large velocity gradients → $\frac{d\vec{v}}{dt} = \frac{\partial\vec{v}}{\partial t}$ and friction terms can be neglected

Gravitational Instability

Use simplified hydrodynamic equations (1-4)

1. **Navier-Stokes:** $\rho \frac{d\vec{v}}{dt} = -\rho \nabla U_{grav} - \nabla p$

$\frac{\partial \vec{v}_1}{\partial t} = -\nabla U_{gr.} - \frac{1}{\rho_0} \nabla p$

-only perturbation gives pressure gradient
-no large scale flow: $d\vec{v}/dt = \partial\vec{v}/\partial t$
-only ρ_0 determines scaling factor, $\rho \approx \rho_0$

2. **Poisson:** $\Delta U_{grav} = 4\pi G \rho$

$\Delta U_{gr.} = 4\pi G \rho_1$

-only density perturb. contributes to U_{grav}
-infinitely extended medium gives constant shift

3. **Continuity:** $\frac{\partial \rho}{\partial t} = \nabla(\rho \vec{v}) = 0$

$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \vec{v}_1 = 0$

-only ρ_0 determines scaling factor

Use simplified hydrodynamic equations (1-4)

4. Equation of state:

- Use polytropic equation of state
  - Independent of T
  - Only 4 equations needed

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma} \quad \text{or} \quad c_s^2 = \frac{dp}{d\rho} = \gamma \frac{p}{\rho}$$

- $1 + \frac{p_1}{p_0} = \left(1 + \frac{\rho_1}{\rho_0}\right)^{\gamma}$ or $c_s^2 = \gamma \frac{p_0}{\rho_0}$

- **Use with pressure term in Navier-Stokes:**

$$\nabla p_1 = \frac{dp}{d\rho} \nabla \rho_1 = c_s^2 \nabla \rho_1$$

QUIZ

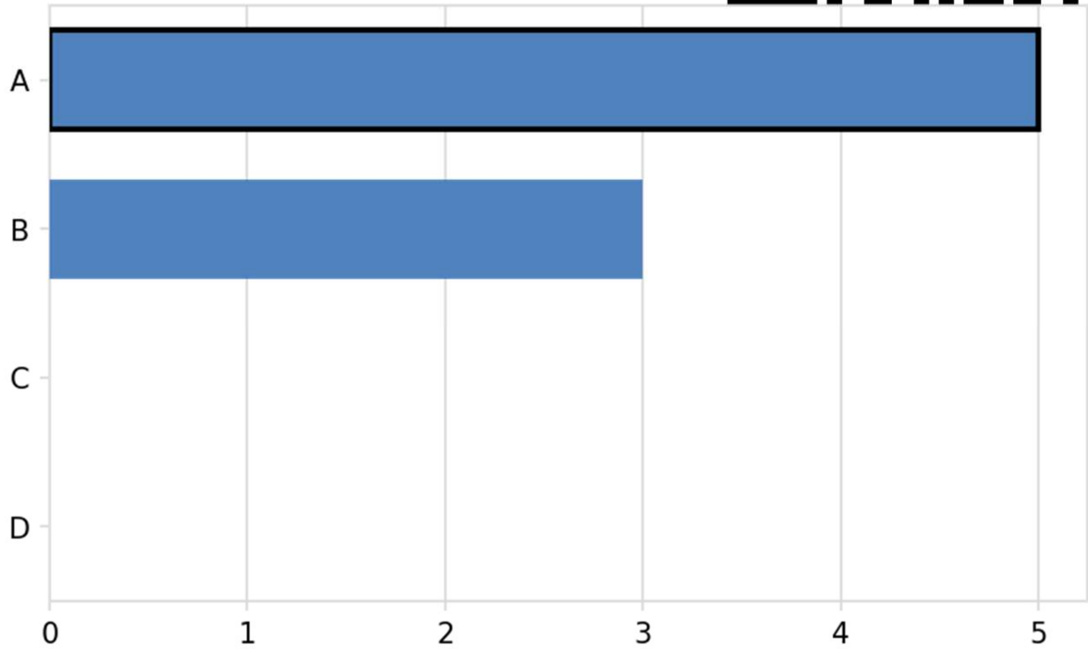
Was bedeutet eine Mach-Zahl $\mathcal{M} \gg 1$?

A) Thermischer Druck ist für die Gasströmung nicht relevant

B) Viskosität dominiert die Strömung

C) Gravitation dominiert

D) Magnetfelder sind unwichtig



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8 Teilnehmer / Umfrage geschlossen

QUIZ

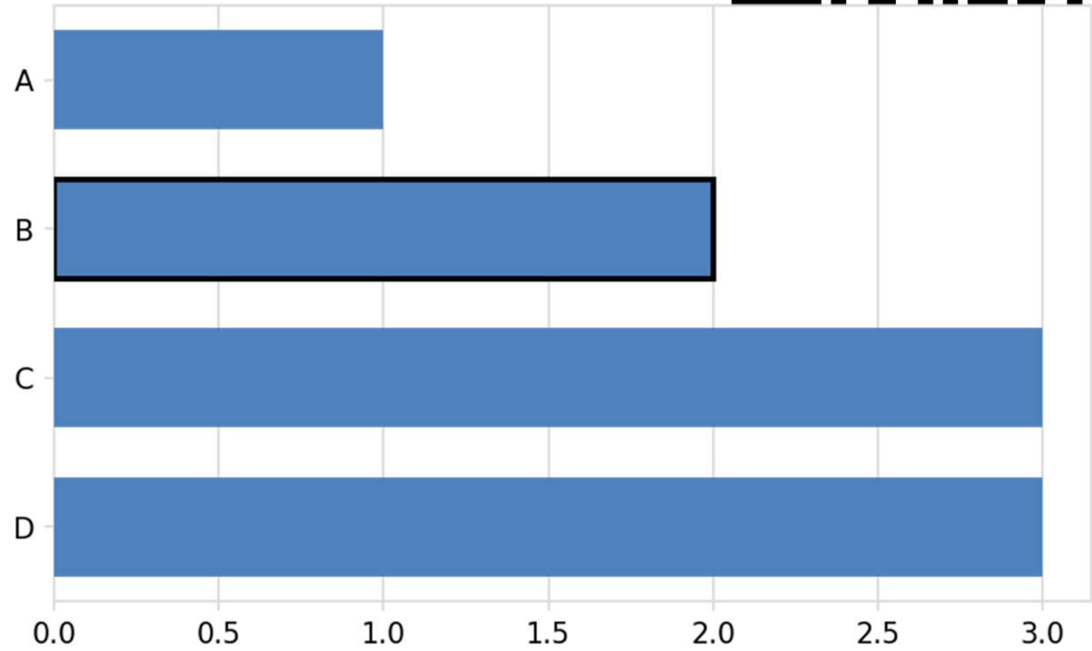
Welche Näherung liegt der „idealen MHD" zugrunde?

A) Verschwindende Leitfähigkeit

B) Unendlich hohe Leitfähigkeit

C) Keine Magnetfelder

D) Nicht-relativistische Strömung mit Viskosität



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QUIZ

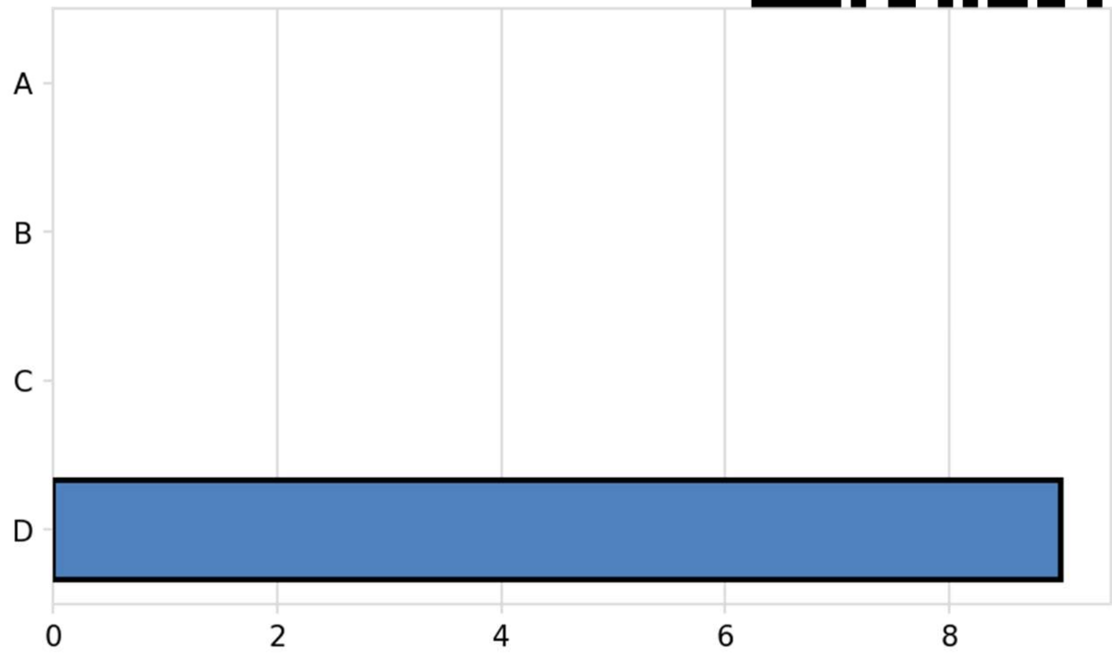
Die Bonnor-Ebert-Sphäre unterscheidet sich von der singulären isothermen Sphäre dadurch, dass…

A) sie nicht-isotherm ist

B) sie rotiert

C) sie magnetisiert ist

D) sie einen regulären Kern hat und durch einen äußeren Druckrand abgeschnitten wird



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9 Teilnehmer / Umfrage geschlossen

Combine hydrodynamic equations(1-4)

1. **Navier-Stokes:**

$\nabla U_{grav} = -\frac{\partial \vec{v}_1}{\partial t} - \frac{c_s^2}{\rho_0}\nabla \rho_1$

- **Take div:**

$\rho_0 \Delta U_{grav} = -\rho_0 \nabla \frac{\partial \vec{v}_1}{\partial t} - c_s^2 \Delta \rho_1$

- **Put into 2)**

$$\Delta U_{grav} = 4\pi G \rho_1$$

$4\pi G \rho_0 \rho_1 = -\rho_0 \frac{\partial}{\partial t}(\nabla \vec{v}_1) - c_s^2 \Delta \rho_1$

- **Derivative of 3) for t:**

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \vec{v}_1 = 0$$

$$\frac{\partial^2 \rho_1}{\partial t^2} = -\rho_0 \frac{\partial}{\partial t}(\nabla \vec{v}_1)$$

spatial and temporal derivative exchangeable

Gravitational Instability

Combine results:

- **Transformed 2):** $4\pi G\rho_0\rho_1 = -\rho_0 \frac{\partial}{\partial t}(\nabla \vec{v}_1) - c_s^2 \Delta\rho_1$
- **Modified 3)** $\frac{\partial^2 \rho_1}{\partial t^2} = -\rho_0 \frac{\partial}{\partial t}(\nabla \vec{v}_1)$
- **Substitute $\vec{v}_1$ term:** $\frac{\partial^2 \rho_1}{\partial t^2} = c_s^2 \Delta\rho_1 + 4\pi G\rho_0\rho_1$

=modified wave equation for ρ_1

- **Solution by wave ansatz:** $\rho_1 = c\exp\left(i\left[\vec{k}\vec{r} - \omega t\right]\right)$
  - represents waves in direction $\vec{k}$
  - full solution=superposition of all possible waves
  - amplitude *c* determined by boundary conditions

Wave solutions:

- **Ansatz:** $\rho_1 = c \exp\left(i\left[\vec{k}\,\vec{r} - \omega t\right]\right)$

  in wave equation: $\frac{\partial^2 \rho_1}{\partial t^2} = c_s^2 \Delta\rho_1 + 4\pi G \rho_0 \rho_1$

  gives $-\omega^2 \rho_1 = -c_s^2 |\vec{k}^2| \rho_1 + 4\pi G \rho_0 \rho_1$

- **Provides the dispersion relation of density waves in the ISM**

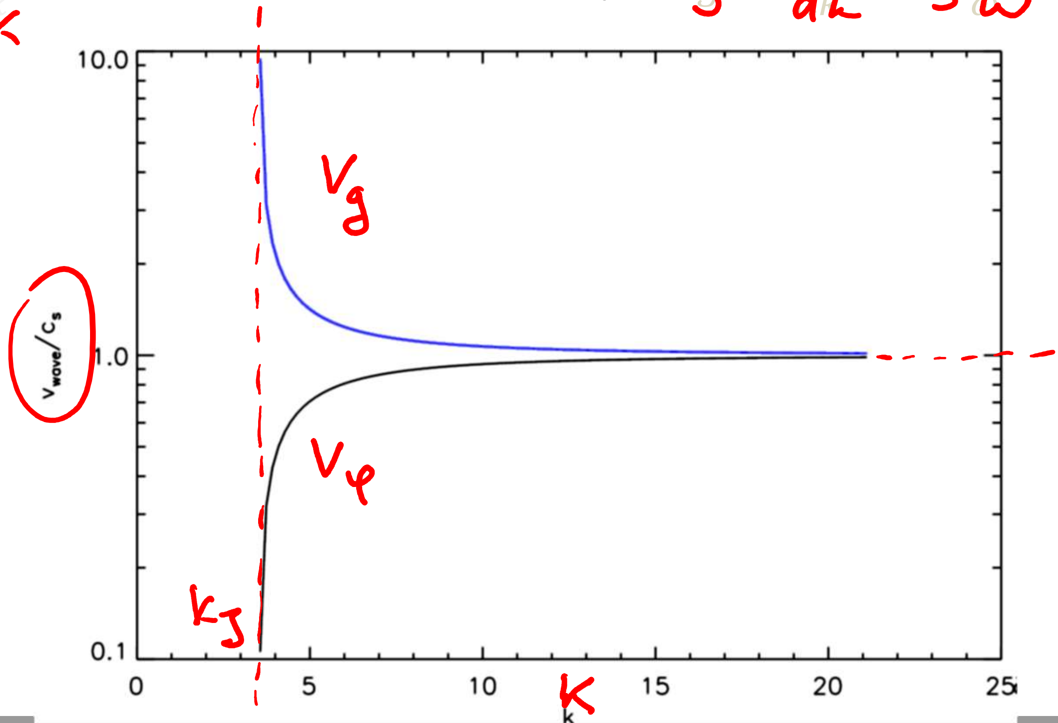
  $\omega^2 = c_s^2 k^2 - 4\pi G \rho_0$

  - relation between wave number $\vec{k}$ and frequency ω
  - isotropic propagation as it only depends on $k = |\vec{k}|$

Wave solutions:

- **Wave propagation velocity (2 types)**

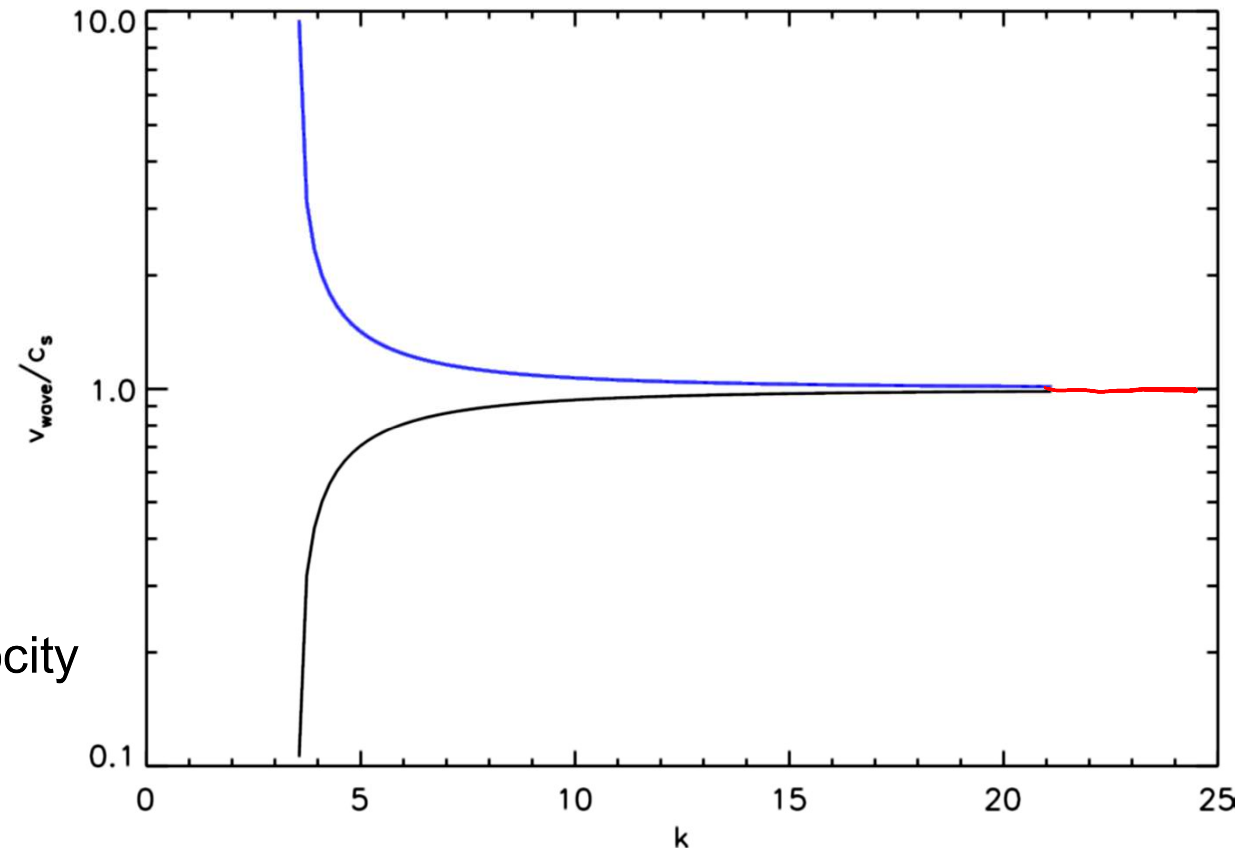
  - Phase velocity: $v_\varphi = \frac{\omega}{k}$
  - Group velocity: $v_g = \frac{d\omega}{dk} = c_s^2 \frac{k}{\omega}$
  - different k-dependence:



Gravitational Instability

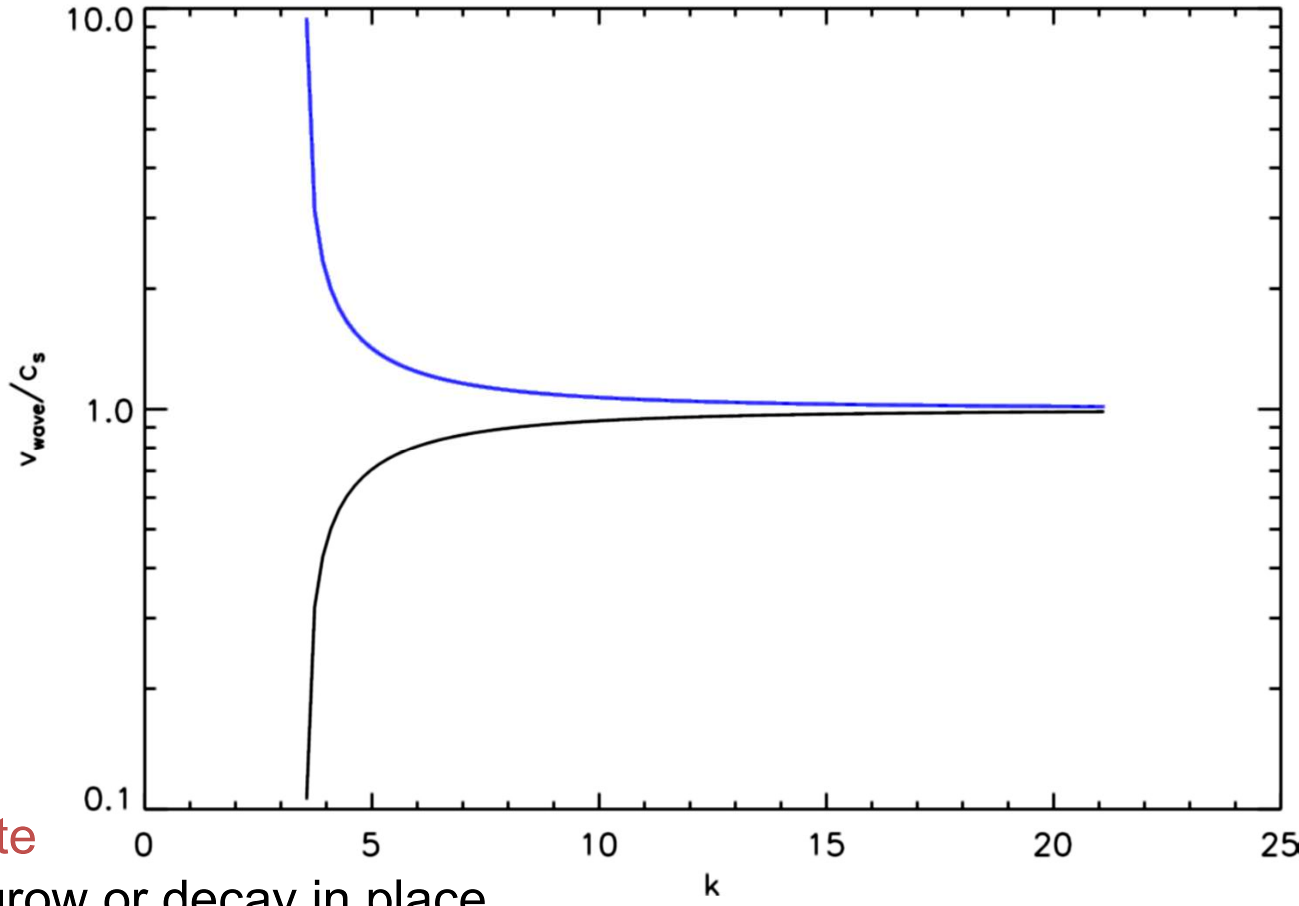
Wave propagation:

- **Limiting cases:**
  - a) ***k*** **large (small wavelengths)**
    - $v_\varphi \approx v_g$
    - no dispersion of the waves
    - waves propagate with c_s
    - Classical sound waves
      - For ideal gas:
        sound speed=thermal velocity
      - $c_s = \sqrt{\frac{\gamma k_B T}{\mu_{mol}}}$
      - Pure pressure waves – gravitation negligible
      - Sound is a solution of hydrodynamic equations!



Wave propagation:

- **Limiting cases:**
  - b) ***k* small (large wavelengths)**
    - $v_\varphi < c_s$
    - $v_g > c_s$
    - significant dispersion
    - Divergence at
    - $k = k_J = \sqrt{\frac{4\pi G \rho_0}{c_s^2}}$
    - Perturbation with $k < k_J$, i.e. larger sizes cannot propagate
    - They do not propagate but grow or decay in place.
    - Only short wavelength perturbations propagate as sound waves



Behavior at small k:

- **For $k < k_J$:**
  - with $\omega^2 = c_s^2 k^2 - 4\pi G \rho_0$
  - ω imaginary!
  - Wave ansatz $\rho_1 = c \exp\left(i\left[\vec{k}\,\vec{r} - \omega t\right]\right)$

    turns into exponential growth or decay

$$\rho_1 = c \exp\left(\pm \frac{t}{\tau_g}\right) \exp\left(i \vec{k}\vec{r}\right)$$

$\pm$ from two possible rooths of $-\omega^2$

with

$$\tau_g = \frac{i}{\omega} = \frac{1}{\sqrt{4\pi G \rho_0 - c_s^2 k^2}} = \frac{1}{c_s \sqrt{k_J^2 - k^2}}$$

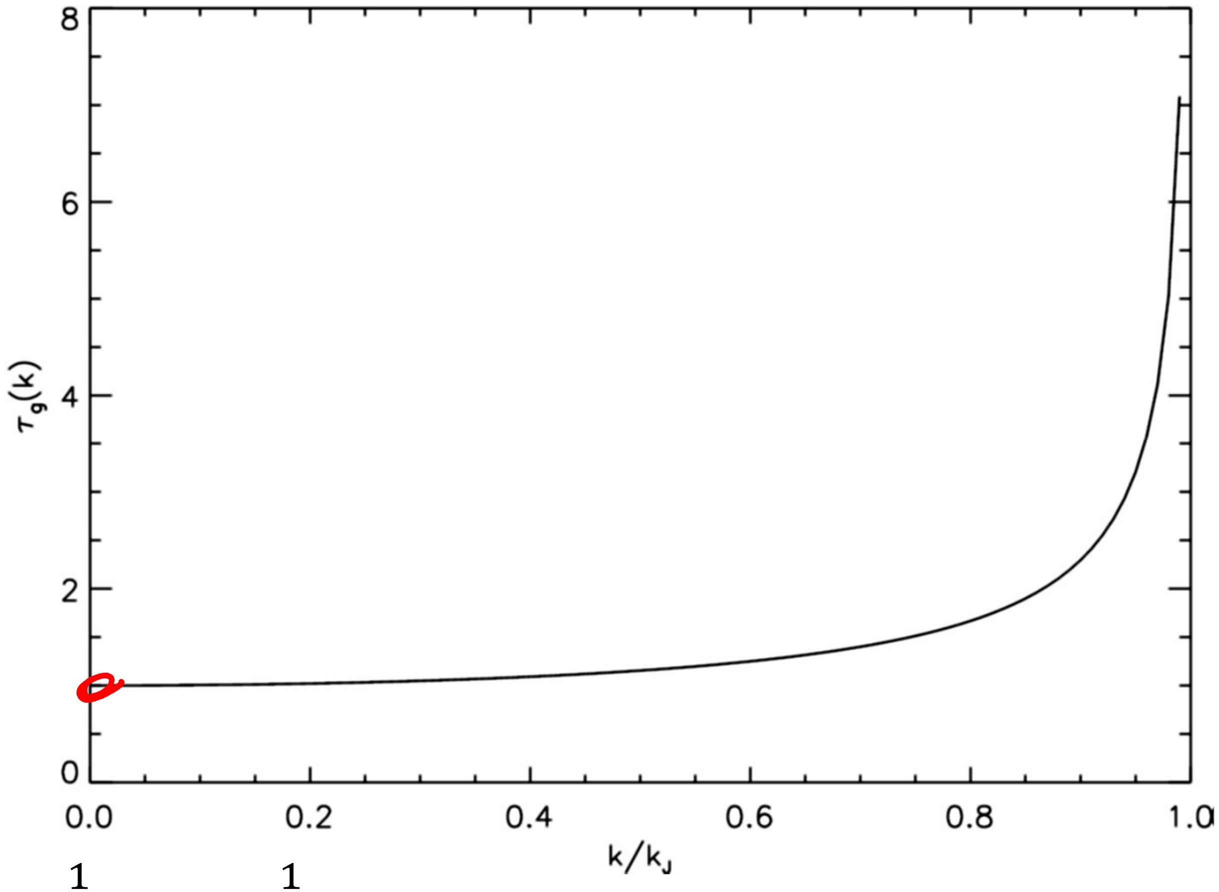
- Exponential growth or exponential damping of density perturbations on timescale τ_g

Gravitational Instability

Exponential solution:

- **Perturbations grow or decay with:**

$$\tau_g = \frac{1}{c_s\sqrt{k_J^2 - k^2}}$$



- General solution = superposition of all possible modes
- After a short time the mode with the fastest growth will „win“, i.e. dominate the structure
- is minimal for $k \rightarrow 0$

$$\tau_0 = \frac{1}{c_s\sqrt{k_J^2}} = \frac{1}{\sqrt{4\pi G\rho_0}}$$

Gravitational Instability

Exponential growth:

- **Dominant mode:** $\rho_1 = c \exp\left(\frac{t}{\tau_0}\right) \times \exp\left(i\vec{k}\,\vec{r}\right)$
- with $\tau_0 = \frac{1}{c_s\sqrt{k_J^2}} = \frac{1}{\sqrt{4\pi G \rho_0}}$ for $k \to 0$
- $k \to 0$ represents infinitely extended perturbations
  - Changes of the overall density are the fastest possible density change!
  - Any extended medium is gravitational unstable
    - will collapse or disperse
    - Stabilization only through boundary conditions:
      - external pressure
      - finite size

Gravitational Instability

Critical size:

- No collapse if modes with $k < k_J$ forbidden by the size of the medium

- **Jeans length:** $$L_J = \frac{2\pi}{k_J} = \sqrt{\frac{\pi c_s^2}{G\rho_0}}$$

- Structures with sizes $> L_J$ are gravitational unstable → will collapse
- Translation into corresponding mass

- **Jeans mass:** $$M_J = \frac{4\pi}{3}\rho_0\left(\frac{L_J}{2}\right)^3 = \frac{\pi}{6}\left(\frac{\pi}{G}\right)^{3/2}\frac{c_s^3}{\sqrt{\rho_0}}$$

- For ideal gas: $$M_J = \frac{\pi}{6}\left(\gamma\frac{\pi k_B}{G\mu_{mol}}\right)^{\frac{3}{2}}\frac{T^{3/2}}{\rho^{1/2}}$$

Gravitational Instability

Jeans mass:

$$M_J = \frac{\pi}{6}\left(\gamma \frac{\pi k_B}{G\mu_{mol}}\right)^{3/2} \frac{T^{3/2}}{\rho^{1/2}}$$

No collapse if mass is $< M_J$

- High T stabilizes
- High ρ destabilizes

Density in particle density: $n = \frac{\rho}{\mu_{mol}}$

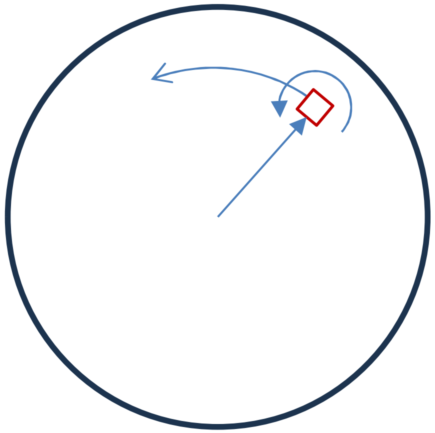
- **Examples**

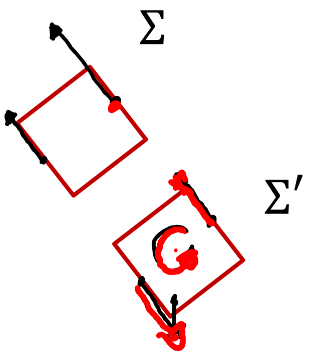
All GMCs should collapse and form stars

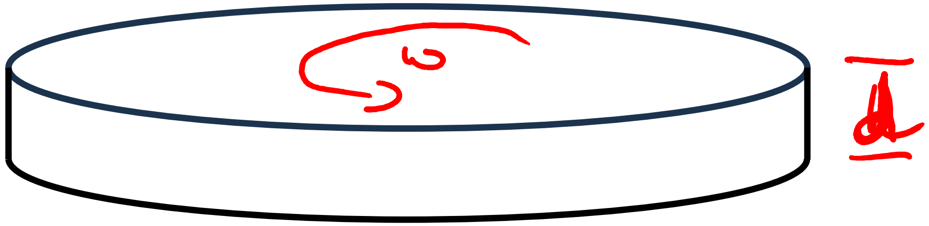
Region	Temperature	Density	Jeans mass	Observed mass
Early universe (after decoupling)	5000 K	100 cm^{-3}	10^6 $M_\odot$	10^{22} $M_\odot$
Diffuse HI clouds	100 K	30 cm^{-3}	10^4 $M_\odot$	10 $M_\odot$
GMCs	10 K	1000 cm^{-3}	10 $M_\odot$	10^3 $M_\odot$

Toomre Instability

- Same approach to analysis, but rotating coordinate system
- Disc description







ω = angular frequency at r

κ = epicyclic frequency $\kappa^2 = \frac{2\omega}{r}\frac{\partial}{\partial r}(r^2\omega)$

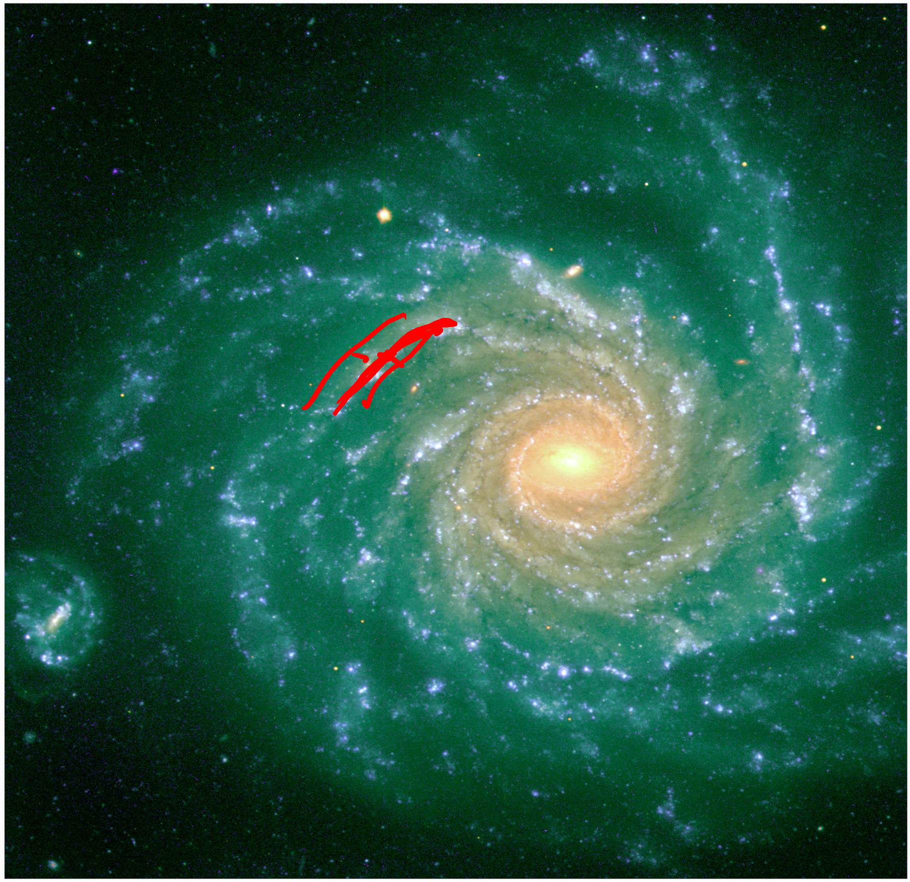
Σ = surface (area) density $\rho d = \Sigma$

- Navier-Stokes equation in co-rotating frame
  - centrifugal force term $f_{cent} = \rho\omega^2\vec{r}$
  - Coriolis force term $f_{coriol} = -2\rho(\vec{\omega}\times\vec{v})$

Gravitational Instability

Toomre-Instability

- Stability criterion
  - **Toomre-parameter**: $Q = \frac{\kappa c_s}{\pi G \Sigma}$
  - For $Q < 1$ axisymmetric perturbations grow exponentially
- **Result**:
  - Spiral arms
  - Inflow of material along arms
  - Transport of angular momentum
  - Formation of
    - GMCs (in a galaxy)
    - Planets (in a protoplanetary system)



Spiral Galaxy NGC 1232 - VLT UT 1 + FORS1

ESO PR Photo 37d/98 (23 September 1998)

QUIZ

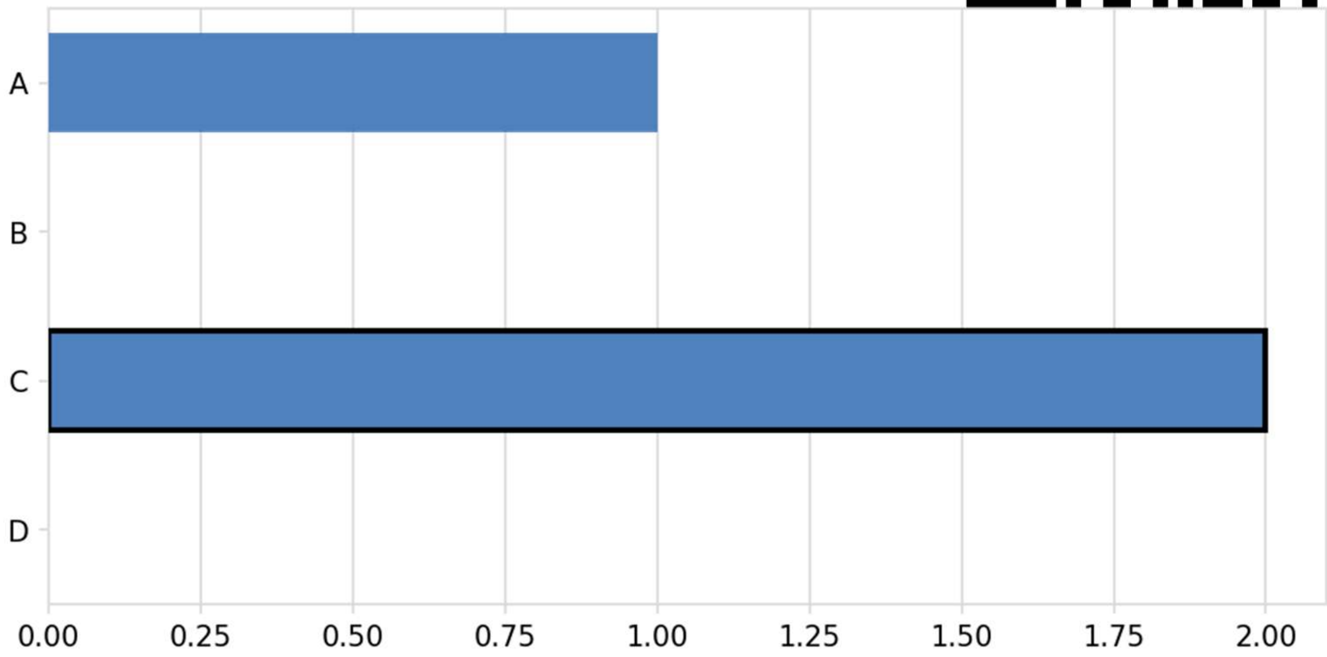
Die Jeans-Länge L_J skaliert wie …

A) $\sqrt{G\rho_0/c_s^2}$

B) $c_s^2 G\rho_0$

C) $\sqrt{c_s^2/(G\rho_0)}$

D) $1/\rho_0$



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3 Teilnehmer / Umfrage geschlossen

QUIZ

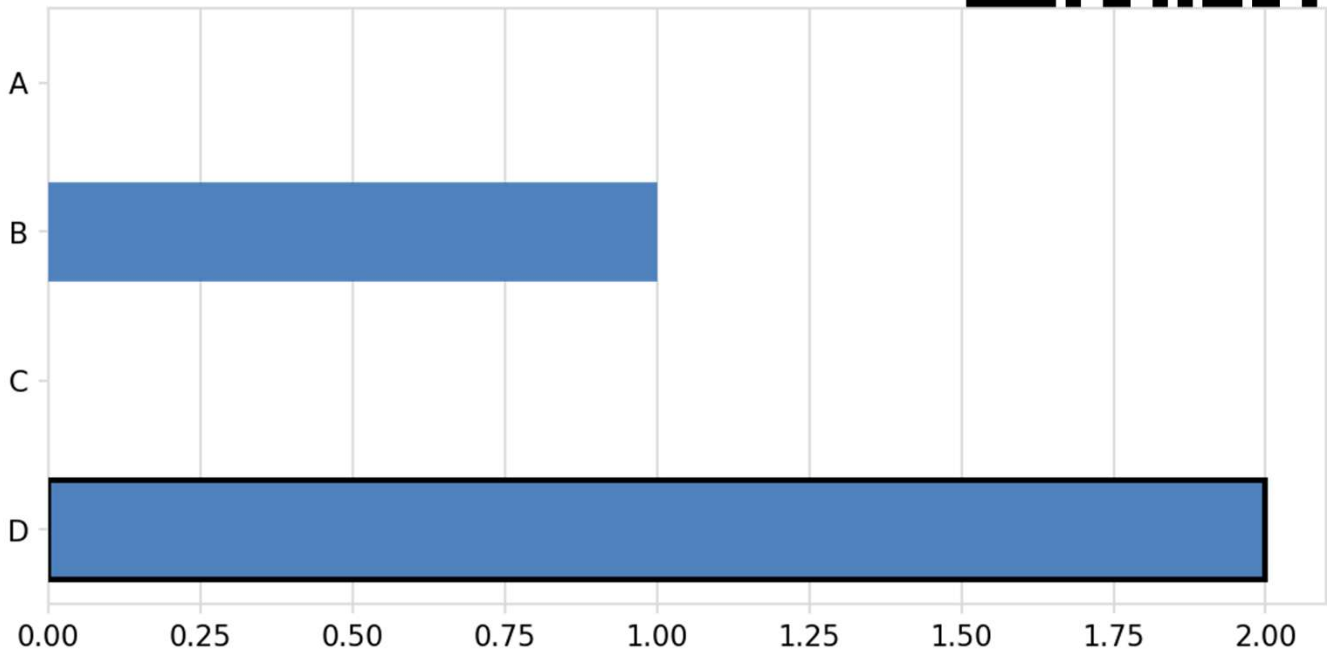
Wie hängt die Jeans-Masse von Temperatur und Dichte ab?

A) $M_J \propto T^{1/2}\rho^{3/2}$

B) $M_J \propto T^{-3/2}\rho^{1/2}$

C) $M_J \propto T\rho$

D) $M_J \propto T^{3/2}\rho^{-1/2}$



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QUIZ

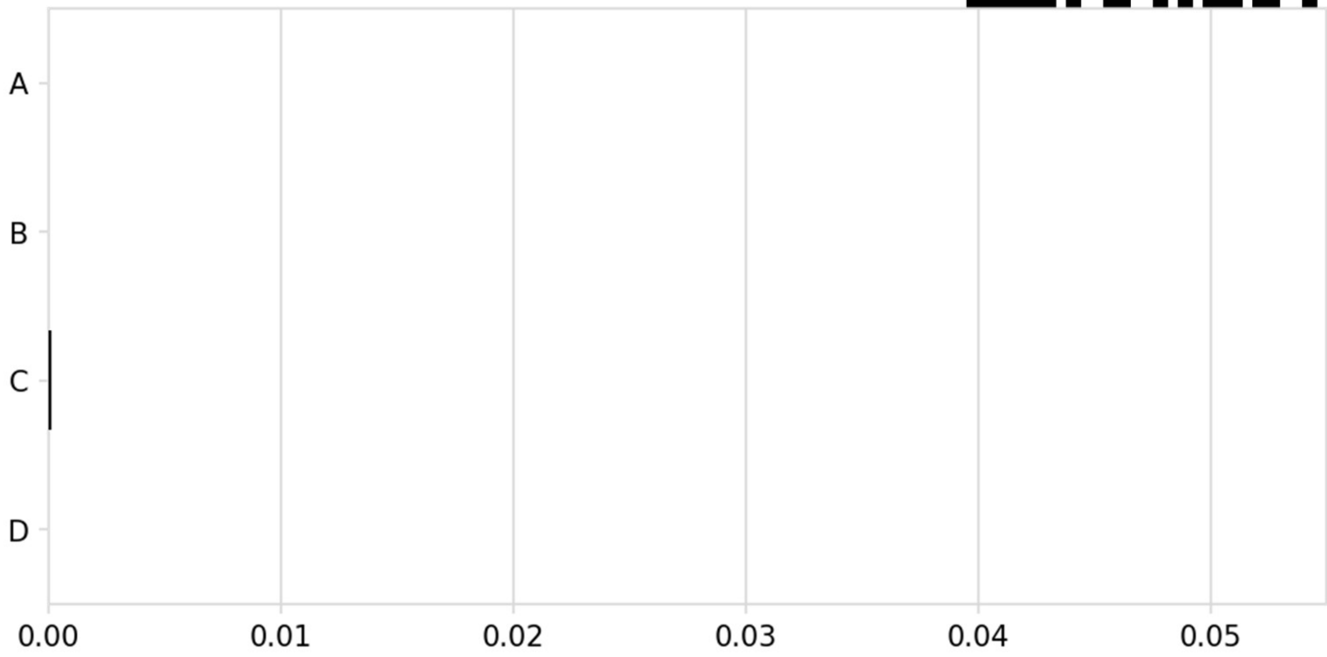
In der Dispersionsrelation der Jeans-Analyse $\omega^2 = c_s^2 k^2 - 4\pi G\rho_0$:Welches k maximiert die Wachstumsrate?

A) $k = k_J$

B) $k \to \infty$

C) $k \to 0$

D) $k = c_s/\sqrt{G\rho_0}$



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QUIZ

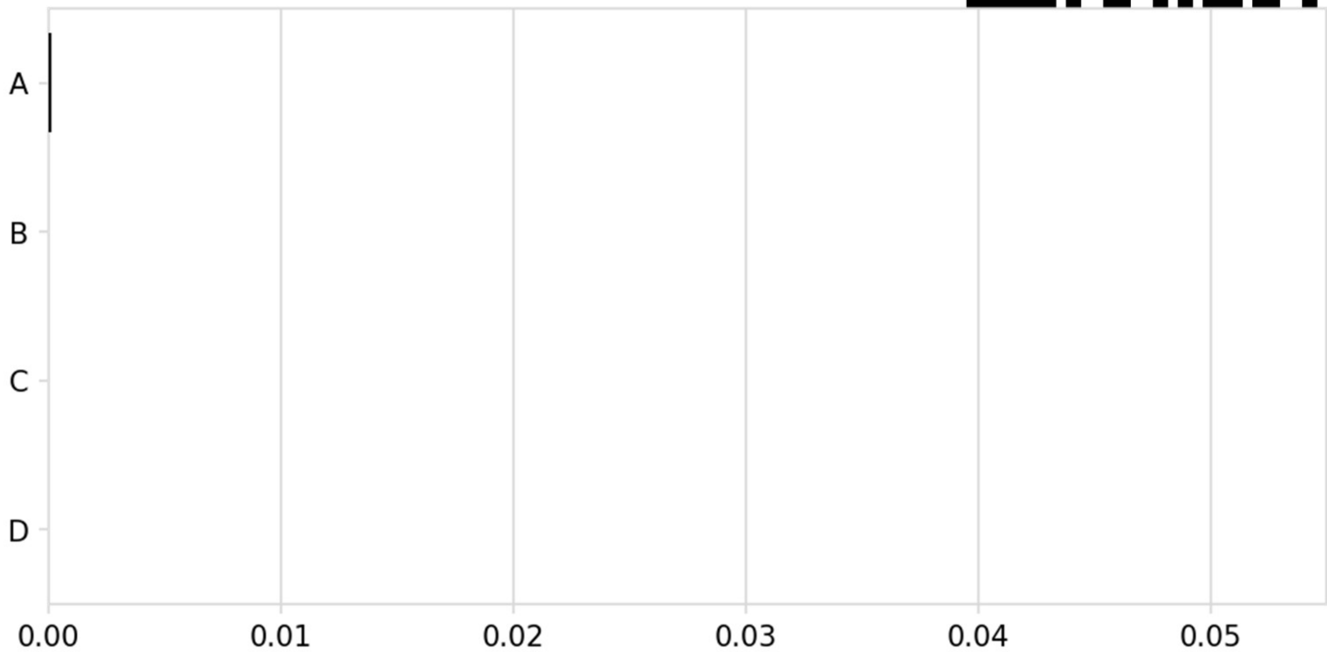
Was ist das Kriterium für die Toomre-Instabilität einer rotierenden Scheibe?

A) $Q = \frac{\kappa c_s}{(\pi G \Sigma)} < 1$

B) $Q > 1$

C) $Q = 0$

D) $Q = \mathcal{M}$



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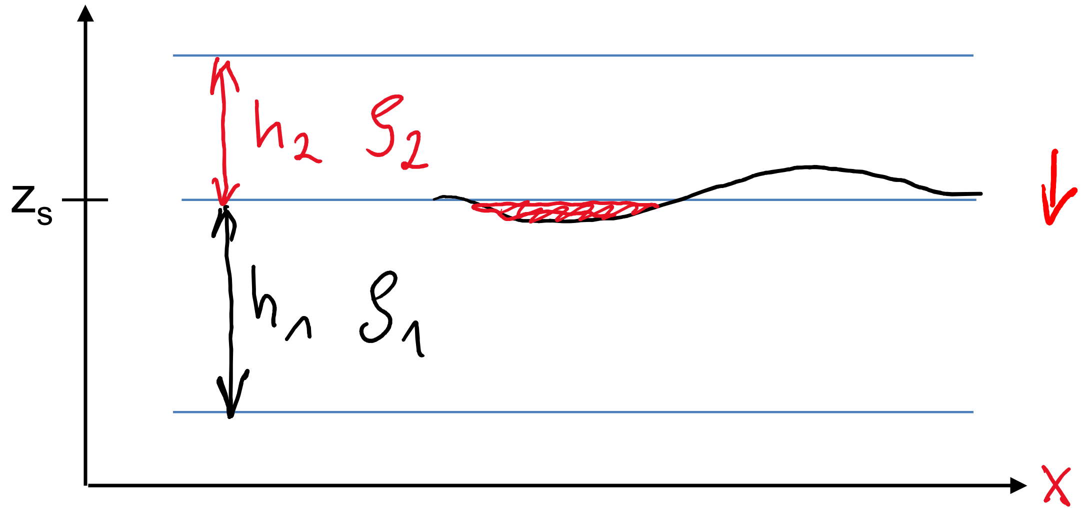
Rayleigh-Taylor Instability

Layering of 2 phases

- **Description**
  - Hydrodynamic equations
  - Gravitation externally dominated: $\vec{F} = \rho \vec{g}$ with $\vec{g} = -\nabla U_{grav}$
  - $\delta e = 0$ (isothermal)

- **Simplification**
  - 2 layers with densities: ρ_1, ρ_2
  - 2-D description

$\rho_2 > \rho_1$



- **Ansatz for perturbation:**
  - Superposition of waves
  - $\delta z(x,t) = c \exp(i[kx - \omega t])$

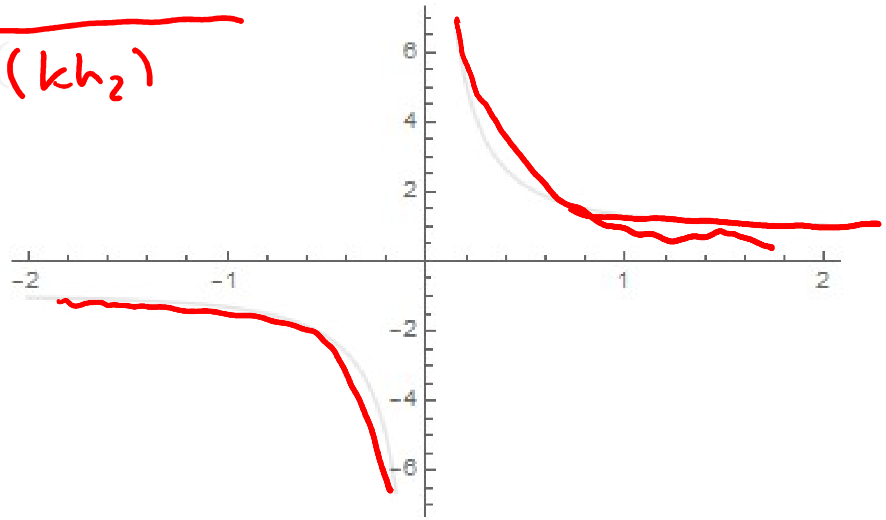
Solution

2

- **Superposition of waves**: $z(x,t) = c \times \exp(i[kx - \omega t])$
  - dispersion relation (without proof)

$$\omega^2 = \frac{kg(\rho_1 - \rho_2)}{\rho_1 \coth(kh_1) + \rho_2 \coth(kh_2)}$$

with $\coth(x) = \frac{e^x + x^{-x}}{e^x - e^{-x}} = \frac{1+e^{-2x}}{1-e^{-2x}}$



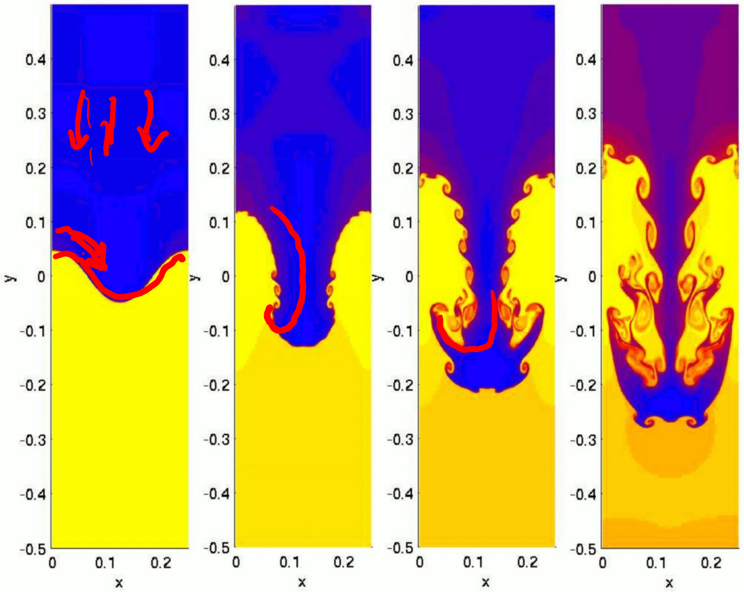
  - Instability if $\omega^2 < 0$,i.e. ω imaginary
    - Happens if $\rho_1 < \rho_2$, i.e. heavy medium on top of light medium
    - **Rayleigh-Taylor instability = inversion instability**

Rayleigh-Taylor Instability

Inversion instability

- **Effect**:
  - Turn-over of material
  - Structure formation at boundary

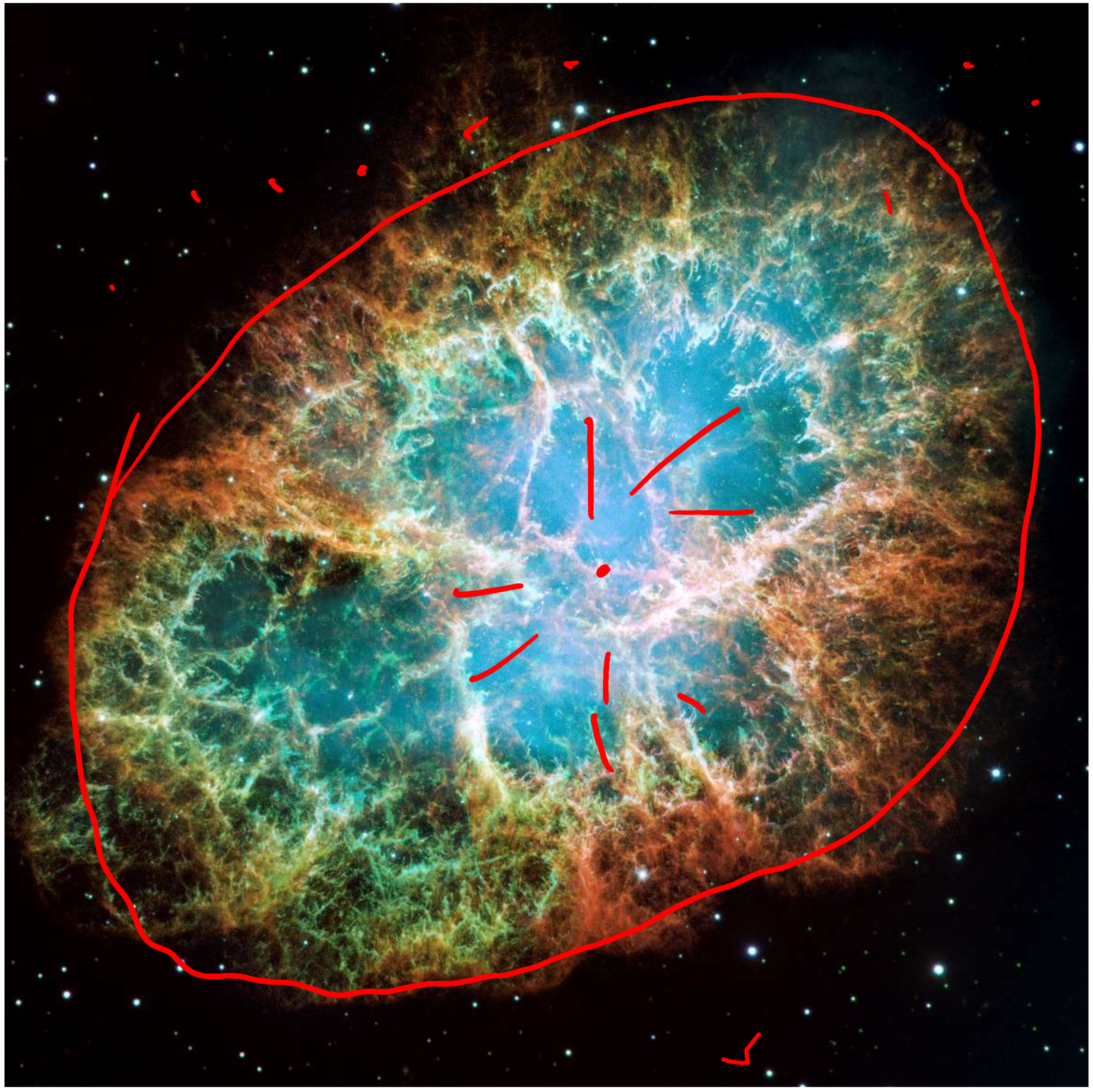
  - Formation of clumps
    → turbulence at boundary
  - instability → torque → creates vorticity
- **MHD Simulation**:
  - „Mushroom cap“ formed
    e.g. by expanding waves



Rayleigh-Taylor Instability

Inversion instability

- **“Effective gravitation”**
  - SN embedded in ISM gas
  - SN ejects shell of stellar material
  - Expanding shell slowed down
  - Equivalence principle:
    - Acceleration $\vec{a}$
    - Equivalent to $\vec{g}$ pointing to center

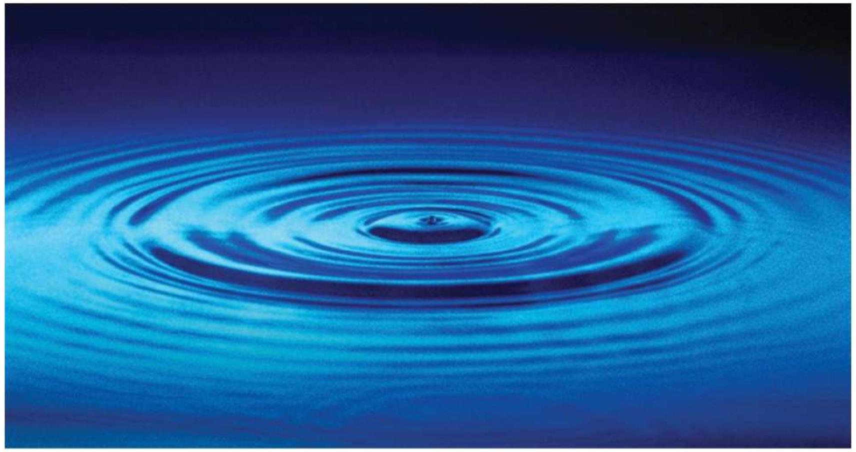


Special (stable) Solution

- **Dispersion relation**: $$\omega^2 = \frac{kg(\rho_1 - \rho_2)}{\rho_1 \coth(kh_1) + \rho_2 \coth(kh_2)}$$

- **Deep-water solution**:
  - „Air“ on „water“: $\rho_1 \gg \rho_2$ → $\rho_1 - \rho_2 \simeq \rho_1$
  - High „atmosphere“: $kh_2 \gg 1$ → $\coth(kh_2) \simeq 1$
  - Deep „water“: $kh_1 \gg 1$ → $\coth(kh_1) \simeq 1$
  - Gives: $\omega^2 \simeq kg$
    - Dispersive: $v_\phi = \frac{\omega}{k} = \sqrt{\frac{g}{k}}$
    - $v_g = \frac{\partial\omega}{\partial k} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{v_\phi}{2}$

  = Gravitational waves on a surface



Special Solution

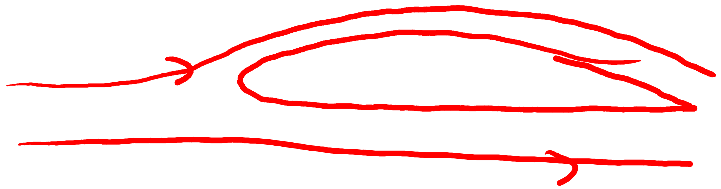
- **Dispersion relation:** $$\omega^2 = \frac{kg(\rho_1 - \rho_2)}{\rho_1 \coth(kh_1) + \rho_2 \coth(kh_2)}$$

- **Shallow-water solution**:
  - „Air" on „water": $\rho_1 \gg \rho_2 \quad \rightarrow \quad \rho_1 - \rho_2 \approx \rho_1$
  - High „atmosphere": $kh_2 \gg 1 \quad \rightarrow \quad \coth(kh_2) \approx 1$
  - Shallow „water": $kh_1 \ll 1 \quad \rightarrow \quad \coth(kh_1) \approx 1/(kh_1)$
  - Gives: $\omega^2 = k^2 g h_1$
    - Linear → not dispersive: $\frac{\omega}{k} = \frac{\partial \omega}{\partial k}$
    - Shallow-water waves travel very far
      - Solitons, tsunamis

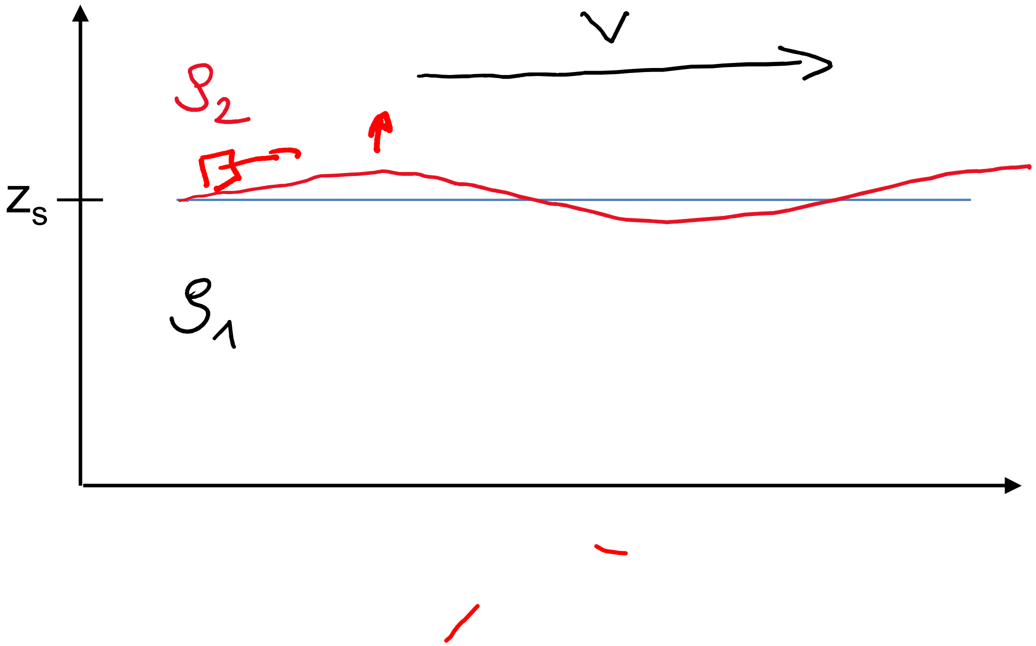
Kelvin-Helmholtz Instability

Moving Layers

- **Same configuration, but**
  - Medium 2 moving relative to medium 1
  - Sign and coordinate system irrelevant as relative motion counts



- **Simplification**
  - $h_1 h_2$ large
  - Neglect gravity
- **Perturbation creates Bernoulli force**
  - Analog to airplane wing
  - v increases at tip of perturbation
  - $p_\perp$ decreases
  - Perturbation grows



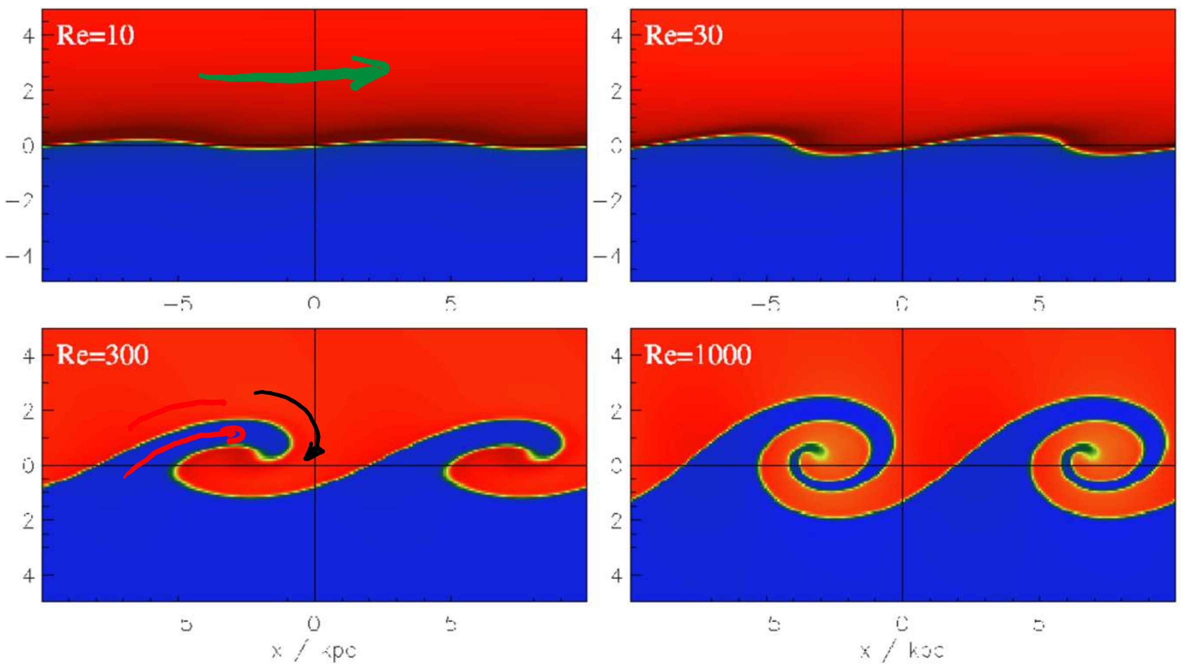
Kelvin-Helmholtz Instability

Wave Ansatz

- **Dispersion relation**:

$$\omega = kv \times \frac{\rho_2 \pm i\sqrt{\rho_1 \rho_2}}{\rho_1 + \rho_2}$$

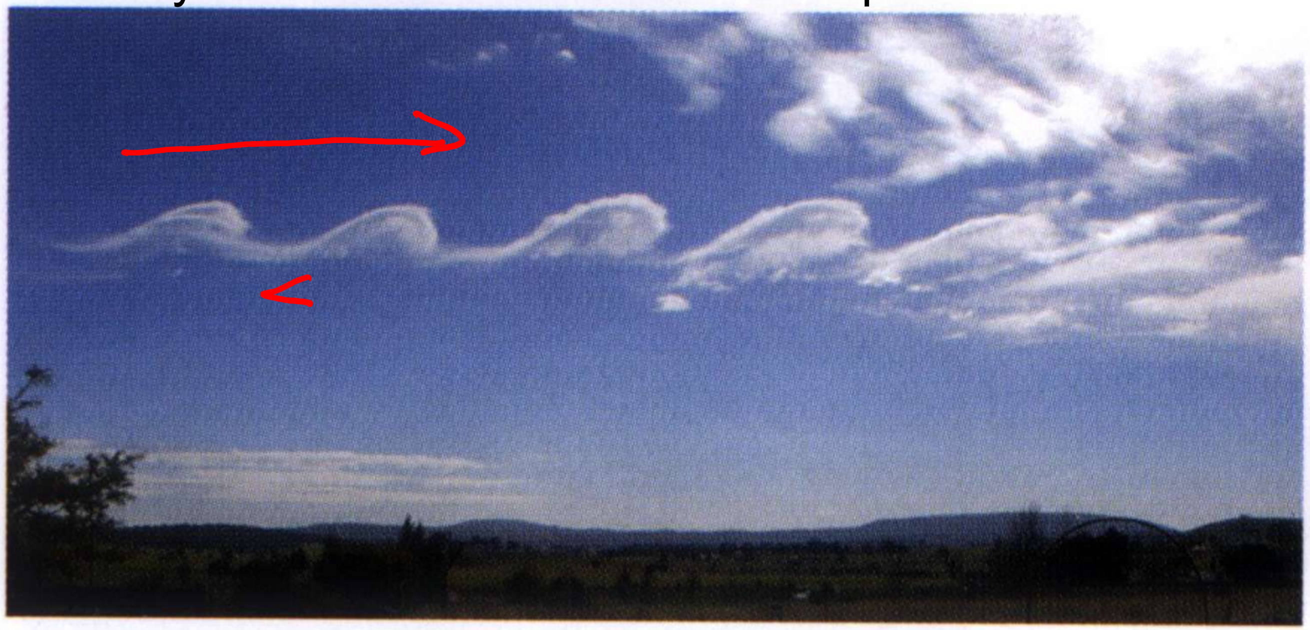
- Always contains imaginary part
- Amplitude of perturbation grows
    - Spatial waves grow in time
    - Temporal waves grow in space
    - $\rightarrow$ **Always unstable!**



HD simulation.
Credit. E. Rödiger(2012)

Sea-wave instability

- **Special case**
  - Shear within a cloud $\rho_1 \approx \rho_2$
  - With disp. relation: $\omega = kv \times \frac{(\rho_2 \pm i\sqrt{\rho_1 \rho_2})}{\rho_1 + \rho_2} \rightarrow$ $\omega = \frac{kv}{2}(1 \pm i)$
  - Every shear in a cloud creates complex structures -> turbulent cascade




Kelvin-Helmholtz Instability

Sea-waves instability

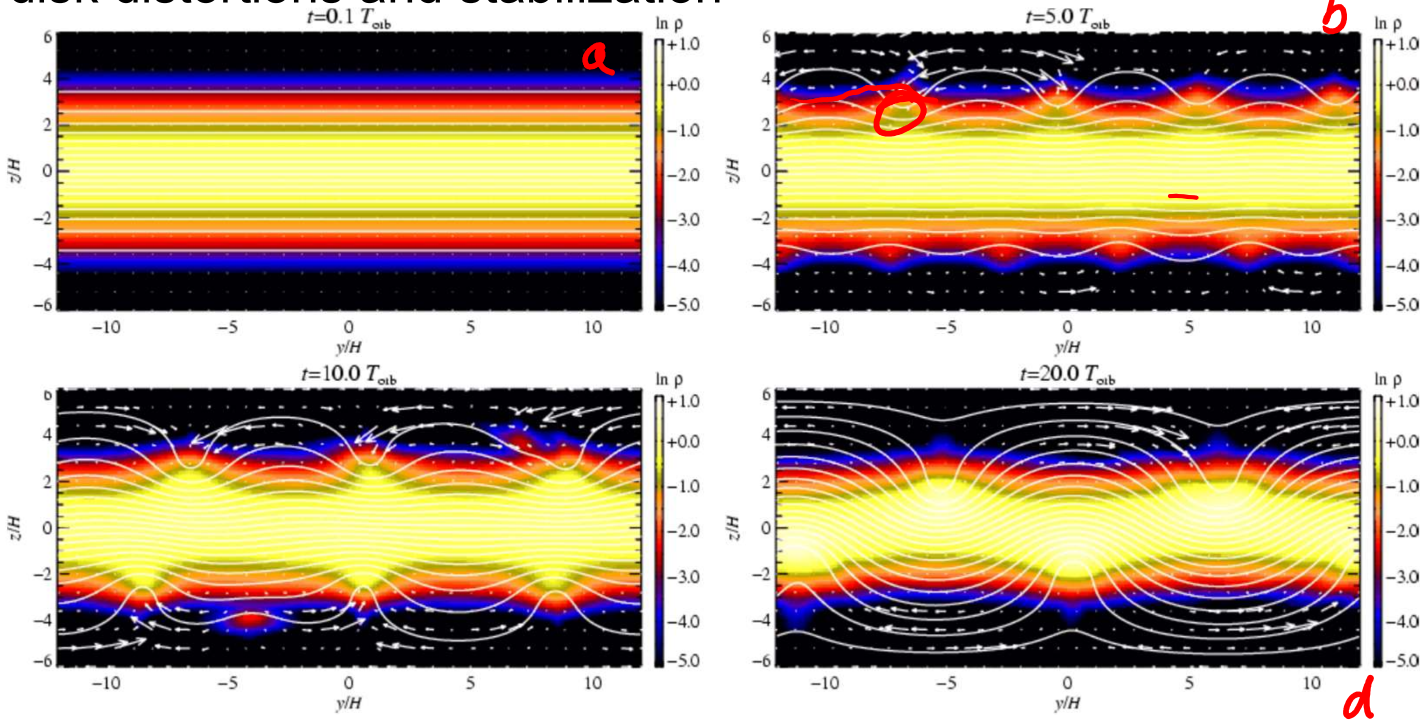
- **Why do not all sea waves grow exponentially?**
  - Add gravity → stabilizes
  - New criterion: Richardson number $R_i = \frac{e_{pot}}{e_{kin}} = -\frac{g}{\rho}\frac{\partial\rho/\partial z}{\rho(\partial v/\partial z)^2}$
    - δz given by the height of the perturbation or actual gradient
  - Instable if $R_i < \frac{1}{4}$
    - Large density gradients stabilize
    - Large velocity gradients de-stabilize

Parker instability

- **MHD effect**
  - Coupling of particles to $\vec{B}$ field lines
  - Creates loops and disks, disk distortions and stabilization

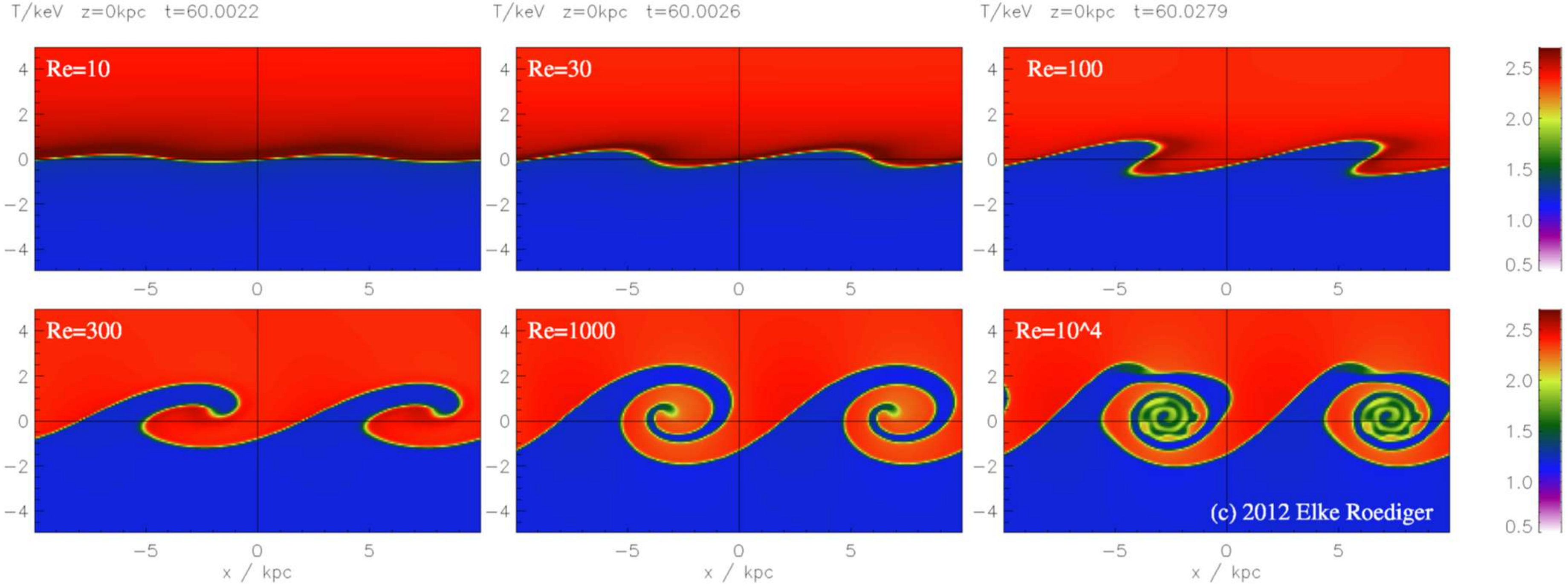
Galaxy

Evolution of a rotating disk structure with parker instability: Johansen & Levin (2008)



Hierarchical structure

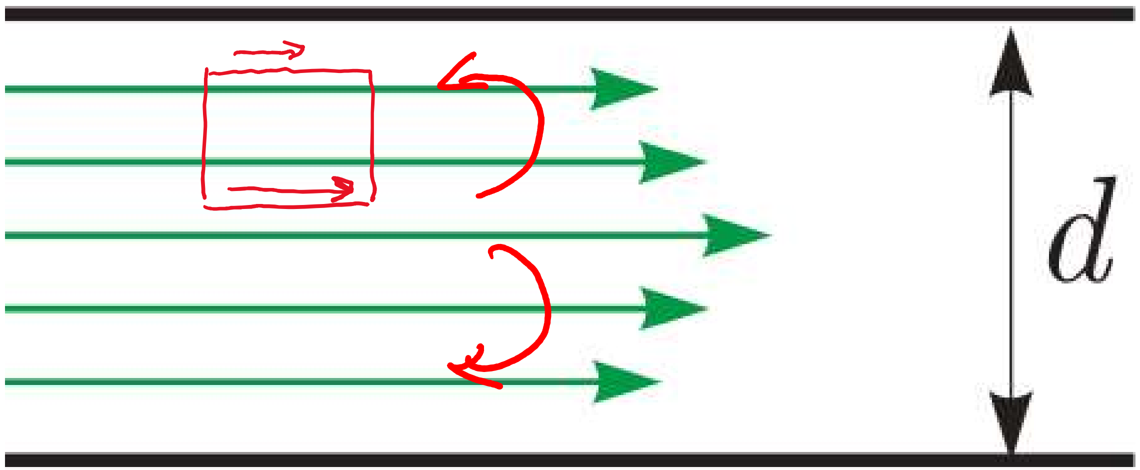
- **Instabilities create structure**
  - Evolution of any instability leads to instabilities on smaller scales
  - Hierarchical structure formation: → Turbulence
- **Example**



Turbulence

Turbulence in the laboratory

- **Flow experiments**:
  - Cascade of eddies Δv



- Criterion for turbulence:
  - Reynolds number $Re := \frac{advection}{dissipation} = \frac{L\Delta v}{\nu} \gg 1$

  - L typical size of the system/driving
  - Δv driving velocity
  - ν kinematic viscosity: $\nu = \frac{\eta}{\rho} = \frac{\lambda c_s}{3}$

η dynamic viscosity

λ mean free path length

Turbulence

Criterion for turbulence

- Reynolds number

$$Re = \frac{L\Delta v}{\nu}$$

Taylor(1964)

- Geometric description of hierarchical structure impossible → Statistical description

Reynolds number

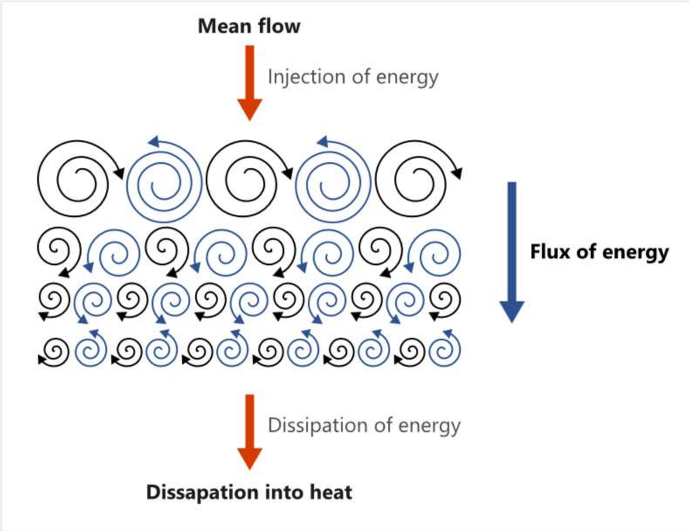
- $Re = \frac{L\Delta v}{\nu} = 3\frac{\Delta v}{c_s}\frac{L}{\lambda} = 3\frac{\Delta v}{c_s}L\sigma_{coll}n$
- **Diffuse interstellar cloud**: $Re \approx 10^5$
- **Molecular cloud**:
  - $L = 0.1pc = 3 \times 10^{17}cm$
  - $\Delta v = 1\ km/s$
  - $T = 20K,\quad \mu_{mol} = \mu_{H_2}\quad \rightarrow c_s = 0.2\ km/s$
  - $n_{H_2} = 10^4 cm^{-3}$ collisional cross section of H_2: $\sigma_{H_2} \approx 2 \times 10^{-16}cm^2$
    - $Re \approx 10^7$
- **The ISM is always turbulent!**
  - Every motion leads to turbulence
  - Every instability induces turbulence

Description of turbulent motions

- Wave description of eddies with different size l
- In Fourier space: $\vec{v}(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int \vec{v}(\vec{r}) e^{i\vec{k}\vec{r}} d\vec{r}$ $\quad k = \frac{2\pi}{l}$
  - Coupling between different waves requires momentum conservation: $\vec{k}_1 + \vec{k}_2 = \vec{k}_3$
  - Two extreme cases:
    - Coupling between modes on same scale $\vec{k}_1$ $\quad |\vec{k}_2| \approx |\vec{k}_1| \quad 2\vec{k}_1 \approx \vec{k}_3$
      - Reduces the eddy scale only by factor 2 (neighboring scale)
      - Energy transfer between neighbor scales → Kolmogorov turbulence
    - Coupling between all scales: $\vec{k}_2 = \int \vec{k}\, d\vec{k}$
      - Shock theory → Burgers‘ turbulence

a) Kolmogorov turbulence (eddy theory)

- **Eddies form energy cascade of motions with different size:**



  - Eddy with size l and velocity v:
    - Energy: $E(l) = \frac{1}{2}\rho l^3 v^2$
    - specific energy density: $e(l) = \frac{v^2}{2}$
  - Energy cascade
    - Injection of motions on larges scale L (velocity V)
    - Dissipation of velocities at small scales by kinematic viscosity ν
    - Transfer of energy from large eddies to smaller eddies

Turbulence

Energy cascade

- **Energy transfer to smaller eddies**
  - At eddy time scale: $\tau_l = l/v$
  - Gives energy transfer rate: $\dot{e} = \frac{1}{2} v^2 \frac{1}{\tau_l} = \frac{1}{2} v^2 \frac{v}{l} = \frac{v^3}{2l}$
- But: energy is not injected at scale l within the cascade, but only injected at scale L and removed at dissipation scale l_s
  - Transfer rate $\dot{e}$ must not depend on l: $\dot{e} = \text{const} \Rightarrow v(l) \sim l^{1/3}$
- **Normalize to largest scale L:** $v(l) = V\left(\frac{l}{L}\right)^{1/3}$ Kolmogorov scaling law
  - v decays towards smaller eddy sizes
  - Angular eddy velocity $\Omega = \frac{2\pi v}{l} \sim l^{-2/3}$ grows toward smaller eddy sizes
  - Eddy lifetime $\tau_l = \frac{l}{v} = 2\pi/\Omega$ shrinks for smaller eddy sizes

End of energy cascade

- **Termination of cascade at scale l_s by viscosity**
  - There $\dot{e}$ can be removed by viscous heating: $\dot{e} = \frac{v}{\tau}\frac{v}{l} = \nu\left(\frac{v_s}{l_s}\right)^2$
  - From cascade $\dot{e} = \frac{v^3}{2l}$, $v_s = V\left(\frac{l_s}{L}\right)^{1/3}$ and rewrite ν: $\nu = \frac{LV}{\mathcal{R}e}$ Substitute v_s :

$$\frac{V^3}{2l_s}\frac{l_s}{L} = \frac{LV}{\mathcal{R}e}\frac{V^2}{l_s^2}\left(\frac{l_s}{L}\right)^{2/3}$$

$$\rightarrow \frac{L^{4/3}}{l^{4/3}}\frac{1}{\mathcal{R}e} = \frac{1}{2} \qquad \rightarrow l_s = \frac{L}{(\mathcal{R}e/2)^{3/4}}$$

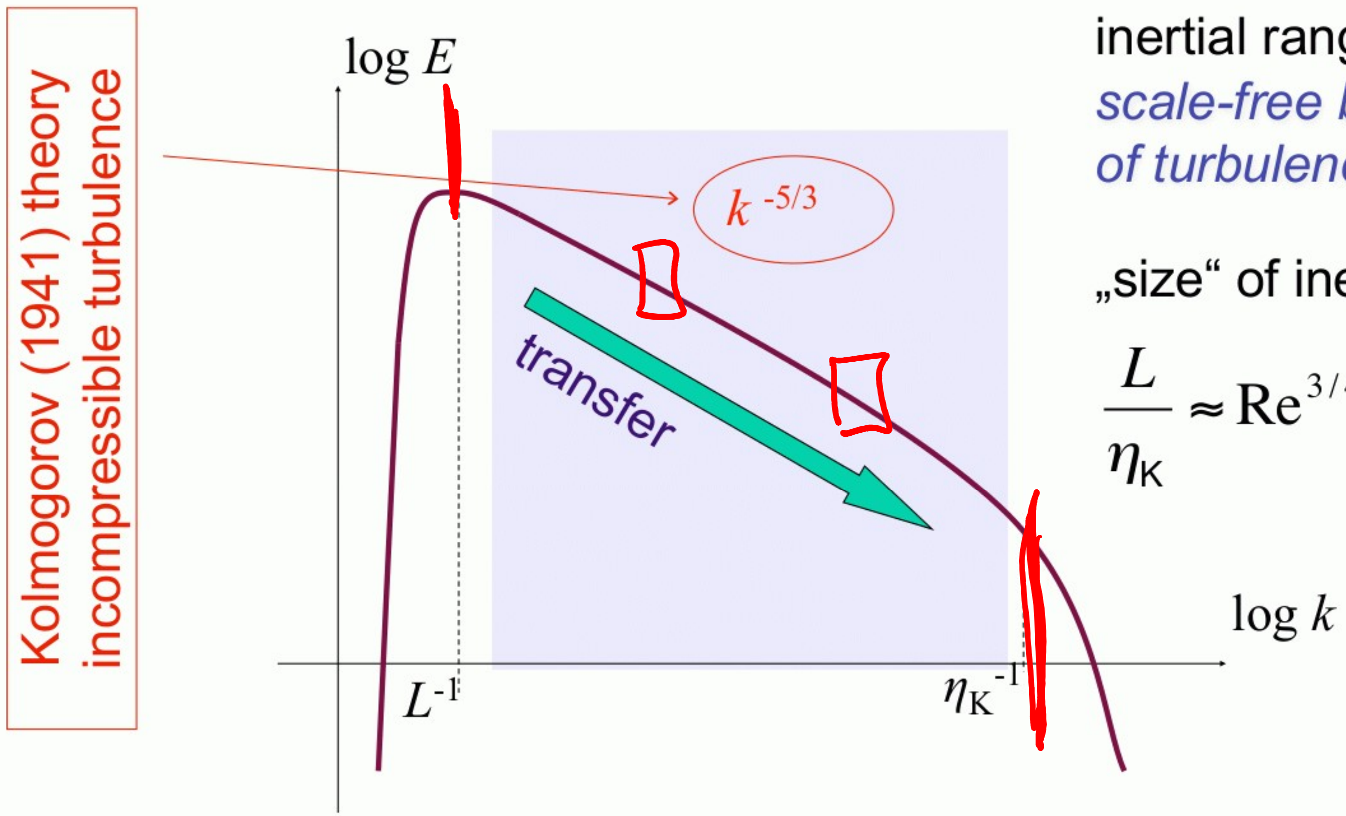
  - The size (length) of the cascade grows almost as fast as the Reynolds number
  - $\mathcal{R}e$ measures the difference in eddy size between injection and dissipation scale

Turbulence

Kolmogorov Spectrum

- $E(k)$ only depends on ϵ $[L^2T^{-3}]$ (energy transfer rate) and k $[L^{-1}]$
- **Dimensional analysis:** $E(k)$ has dimension [energy/mass/k] = $[L^3T^{-2}]$
  - Ansatz: $E(k) = C\,\epsilon^a k^b$ $\qquad [L^3T^{-2}] = [L^{2a}T^{-3a}][L^{-b}]$
  - Therefore $a = 2/3$ and $b = -5/3$
- **Energy spectrum**
  - $$E(k) = C_K \epsilon^{2/3} k^{-5/3}$$
  - Kolmogorov: $E(k) \sim k^{-5/3}$ =Kolmogorov spectrum
- Measures the power in a velocity field as function of the spatial wavenumber

Turbulence



energy
input
scale

energy
dissipation
scale

inertial range:
scale-free behavior of turbulence

„size“ of inertial range:

$$\frac{L}{\eta_K} \approx \mathrm{Re}^{3/4}$$

from R.Klessen (2012)

Turbulence

Observations

- Chaotic nature of turbulent field does not allow to identify individual eddies
  - Measure statistics

- **Statistics of velocity dispersion in clouds:** Size-Linewidth relation
  - Larson (1981):
  - $\sigma(km\ s^{-1}) = 1.10\ L(pc)^{0.38}$
  - Compare Kolmogorov $\sigma \propto L^{0.33}$
  - Somewhat steeper

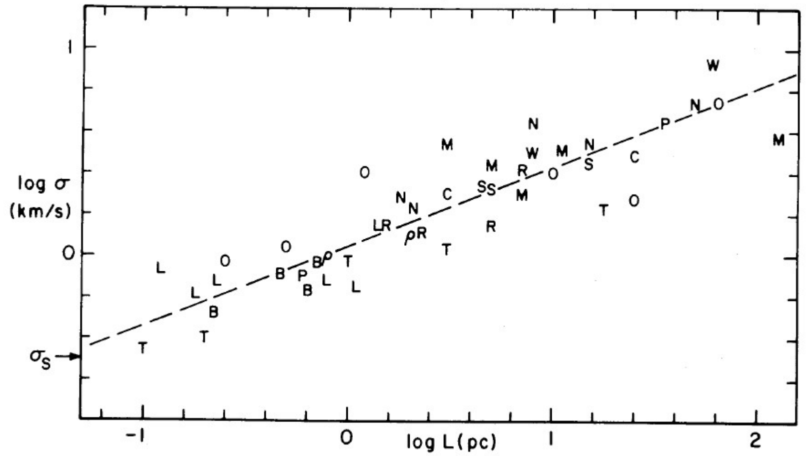


Figure 1. The three-dimensional internal velocity dispersion σ plotted versus the maximum linear dimension L of molecular clouds and condensations, based on data from Table 1; the symbols are identified in Table 1. The dashed line represents equation (1), and σ_S is the thermal velocity dispersion.

Turbulence Observations

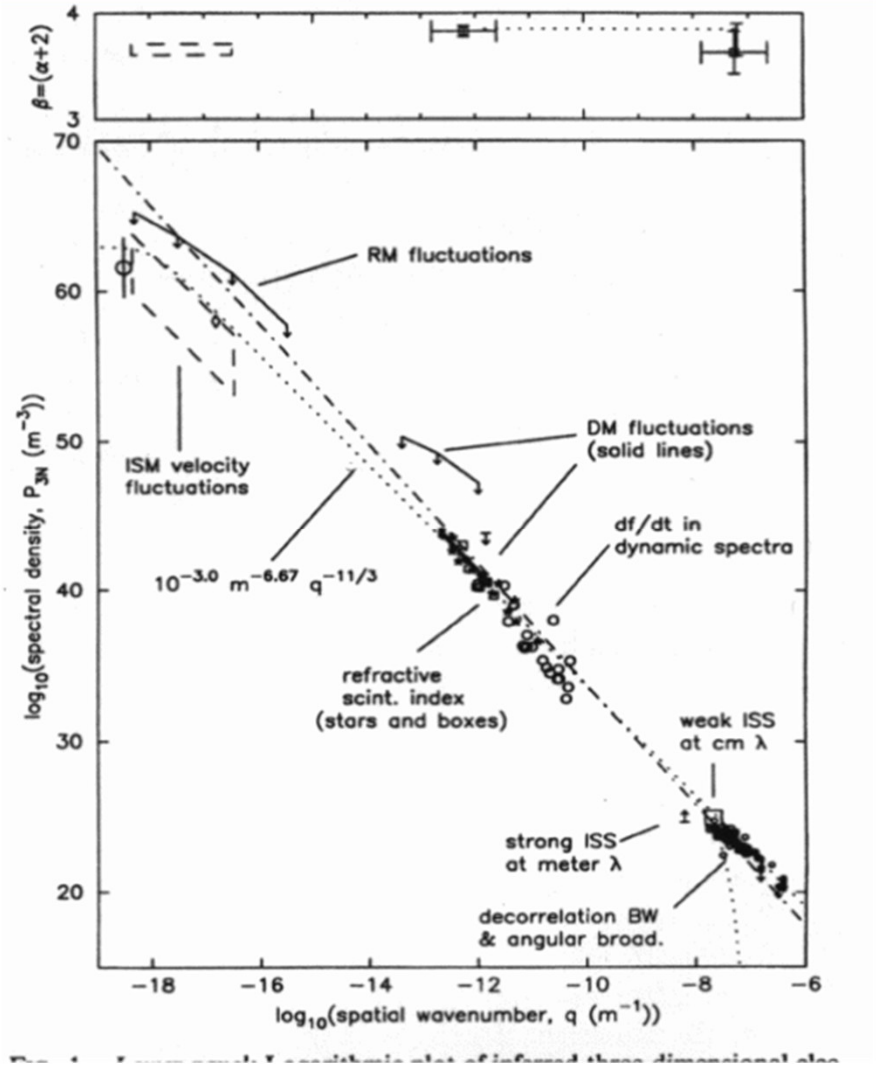
Kolmogorov spectrum

- **Established over a wide range of scale**
  - Exponent: -11/3=-3.67

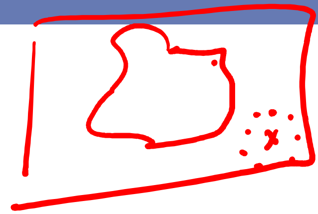
$$E(k) = 4\pi k^2 P(k)$$

But: Kolmogorov should only hold for incompressible turbulence
(no density fluctuations considered)

Armstrong, Ricket & Spangler (1995)



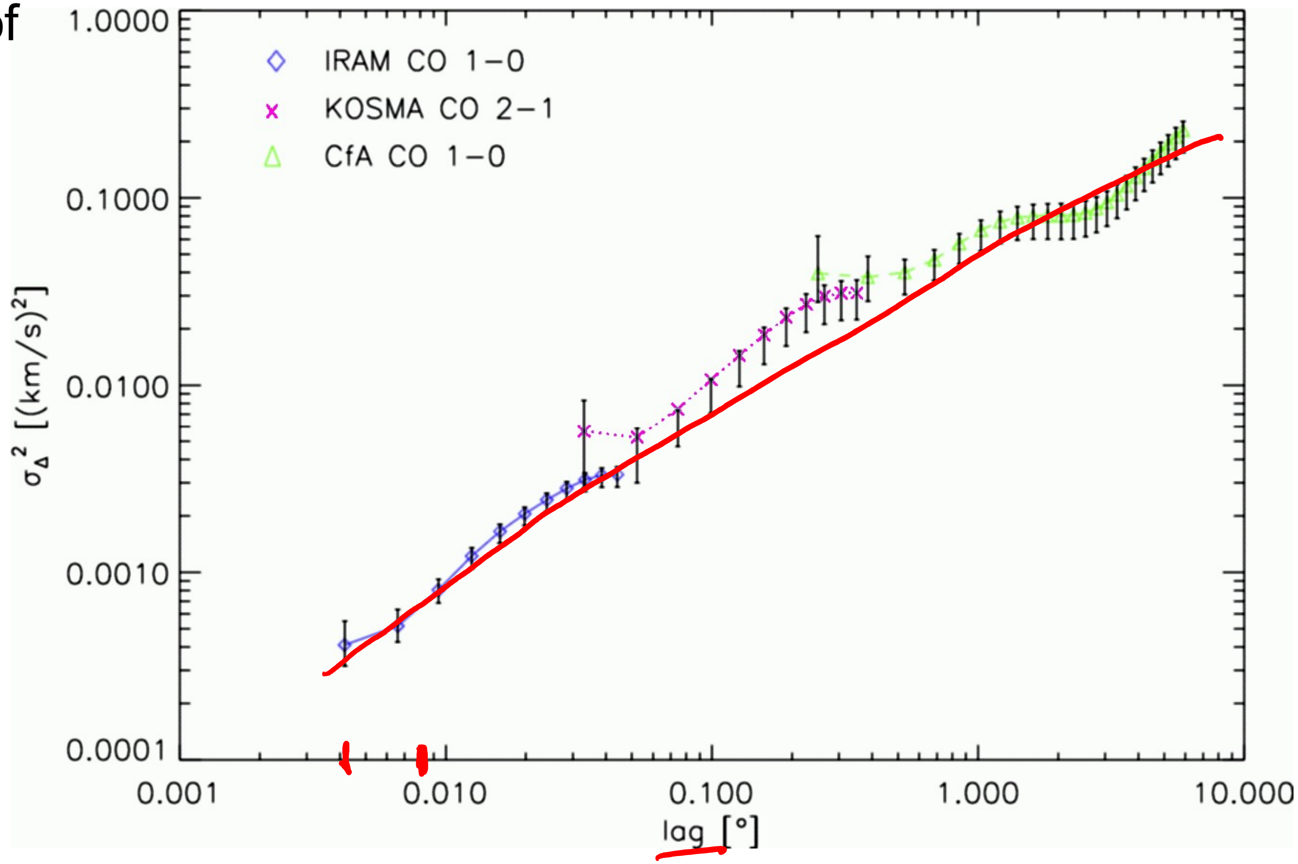
Turbulence observations



Improved velocity measurements

- **Generalized two-point statistics**
  - Velocity differences as function of spatial separation
  - Ossenkopf & McLow (2002):

$\Delta v(\ell) \sim \ell^{1/2}$



  - Even steeper
    - Not consistent with Kolmogorov! velocity field in Polaris flare

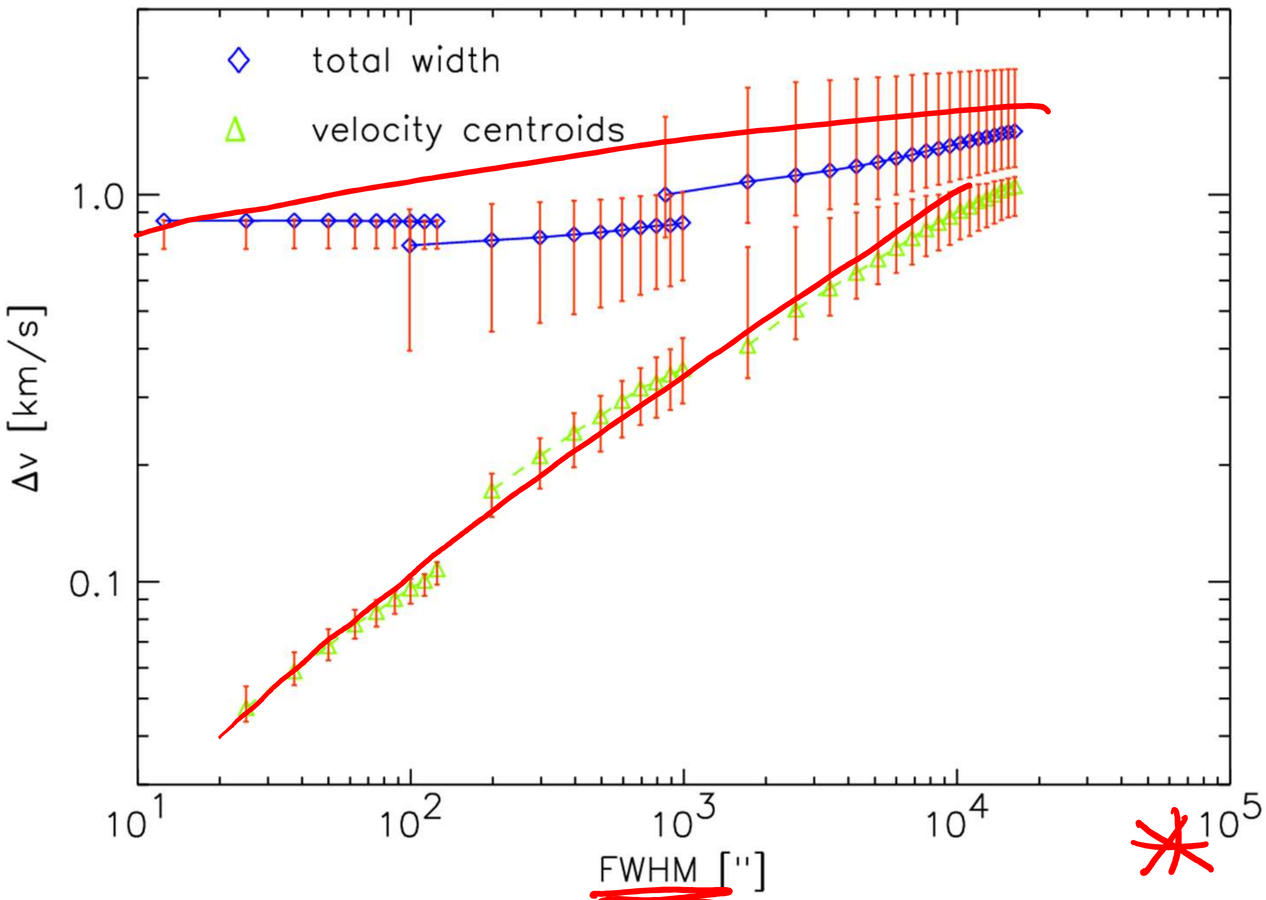
Turbulence observations

Improved velocity measurements

- **Generalized two-point statistics**
  - Velocity differences as function of spatial separation
  - Ossenkopf & McLow (2002):

    $\Delta v(l) \sim l^{1/2}$

  - Applicable to scaling within clouds
  - Large dynamic range with virtual beams



QUIZ

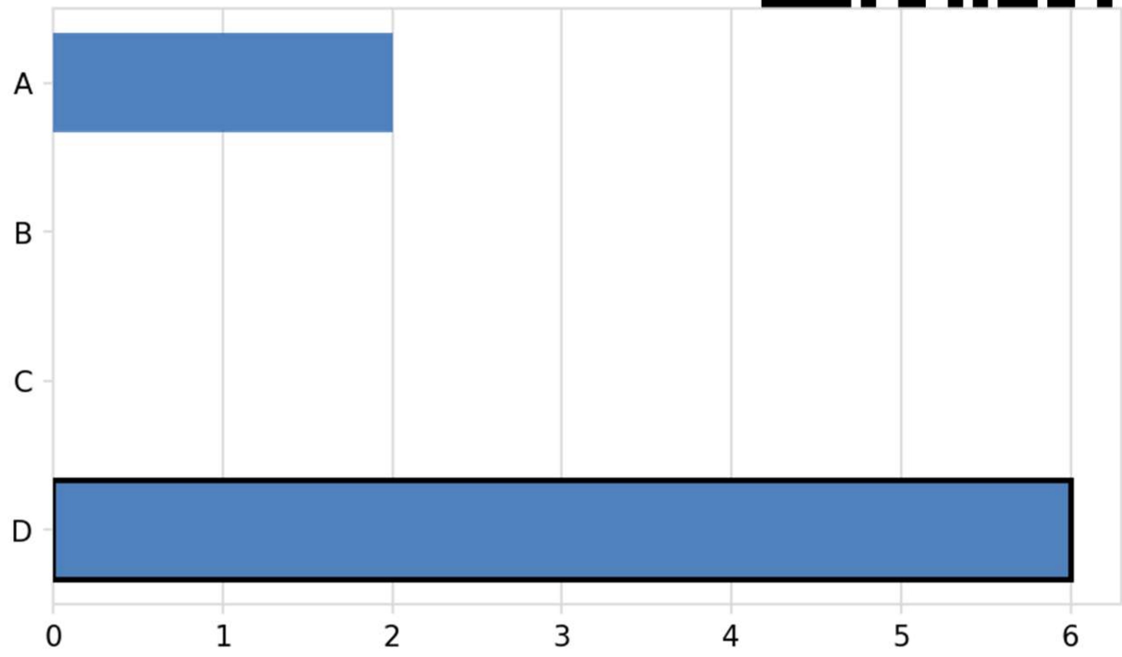
Unter welcher Bedingung tritt die Rayleigh-Taylor-Instabilität auf?

A) Leichtes Medium auf schwerem Medium

B) Zwei Medien gleicher Dichte

C) Nur bei Scherströmungen

D) Schweres Medium auf leichtem Medium



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8 Teilnehmer / Umfrage geschlossen

QUIZ

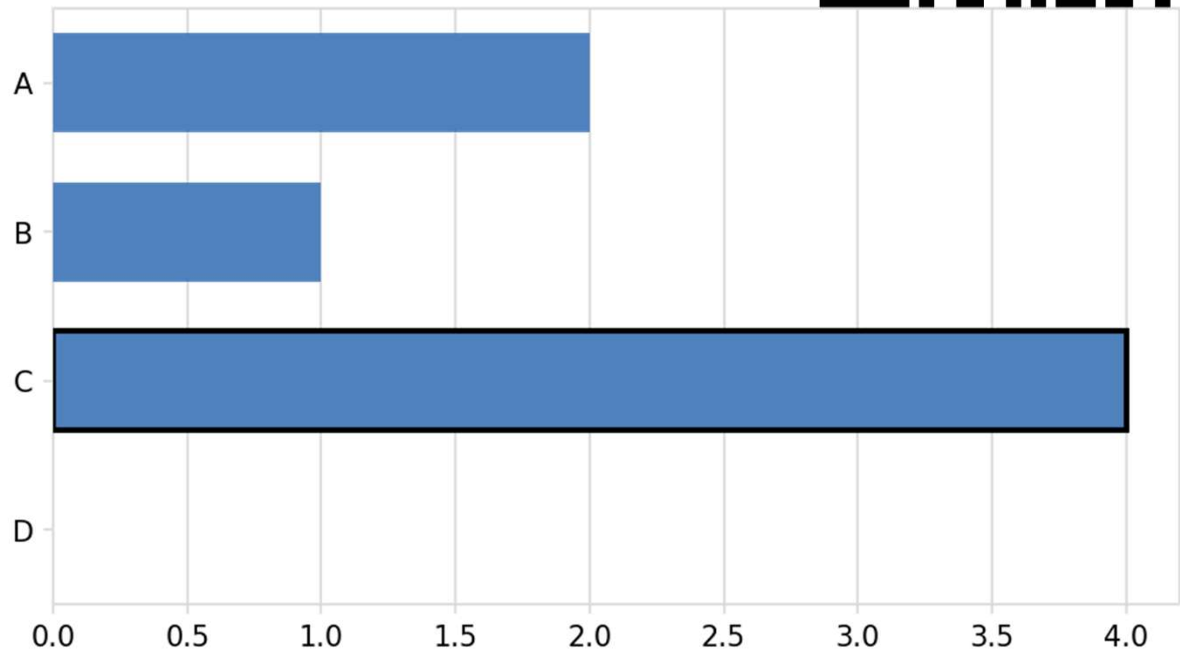
Welche Instabilität ist für jede Scherströmung zwischen zwei Fluidschichten relevant?

A) Jeans-Instabilität

B) Rayleigh-Taylor-Instabilität

C) Kelvin-Helmholtz-Instabilität

D) Parker-Instabilität



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7 Teilnehmer / Umfrage geschlossen

QUIZ

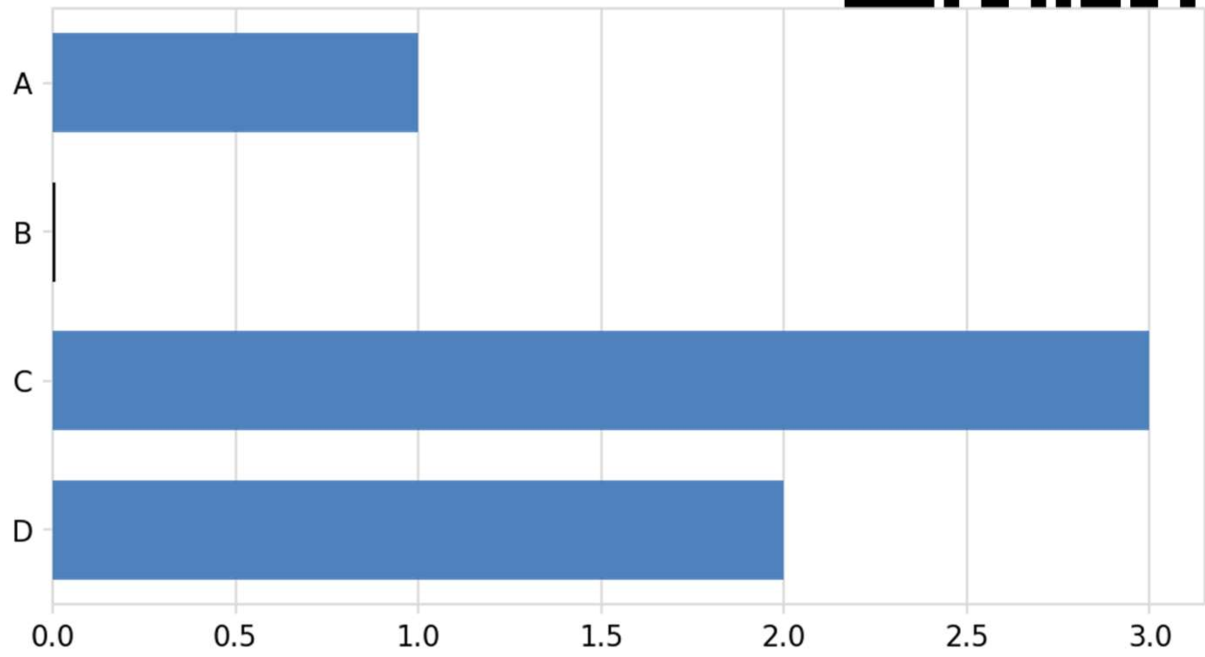
Warum führt die Rayleigh-Taylor-Instabilität in einer expandierenden, abgebremsten SN-Schale zur Fragmentierung?

A) Wegen Coriolis-Kraft

B) Wegen des Äquivalenzprinzips

C) Wegen der Toomre-Instabilität

D) Wegen Kolmogorov-Kaskade



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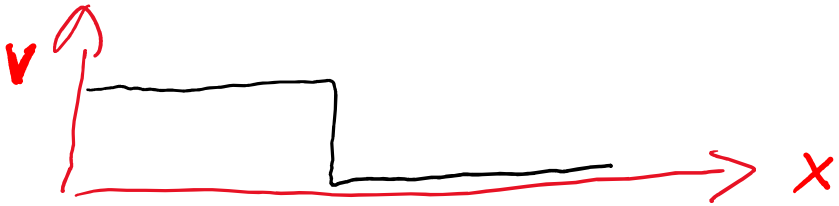
6 Teilnehmer / Umfrage geschlossen

Burgers' turbulence

- **Shocks**
  - velocity profile:



  - **Power spectrum** $P\left(\vec{k}\right) = \frac{1}{(2\pi)^{3/2}} \int \left|\vec{v}\left(\vec{k}\right)\right|^2 e^{i\vec{k}\vec{r}} d^3\vec{r}$

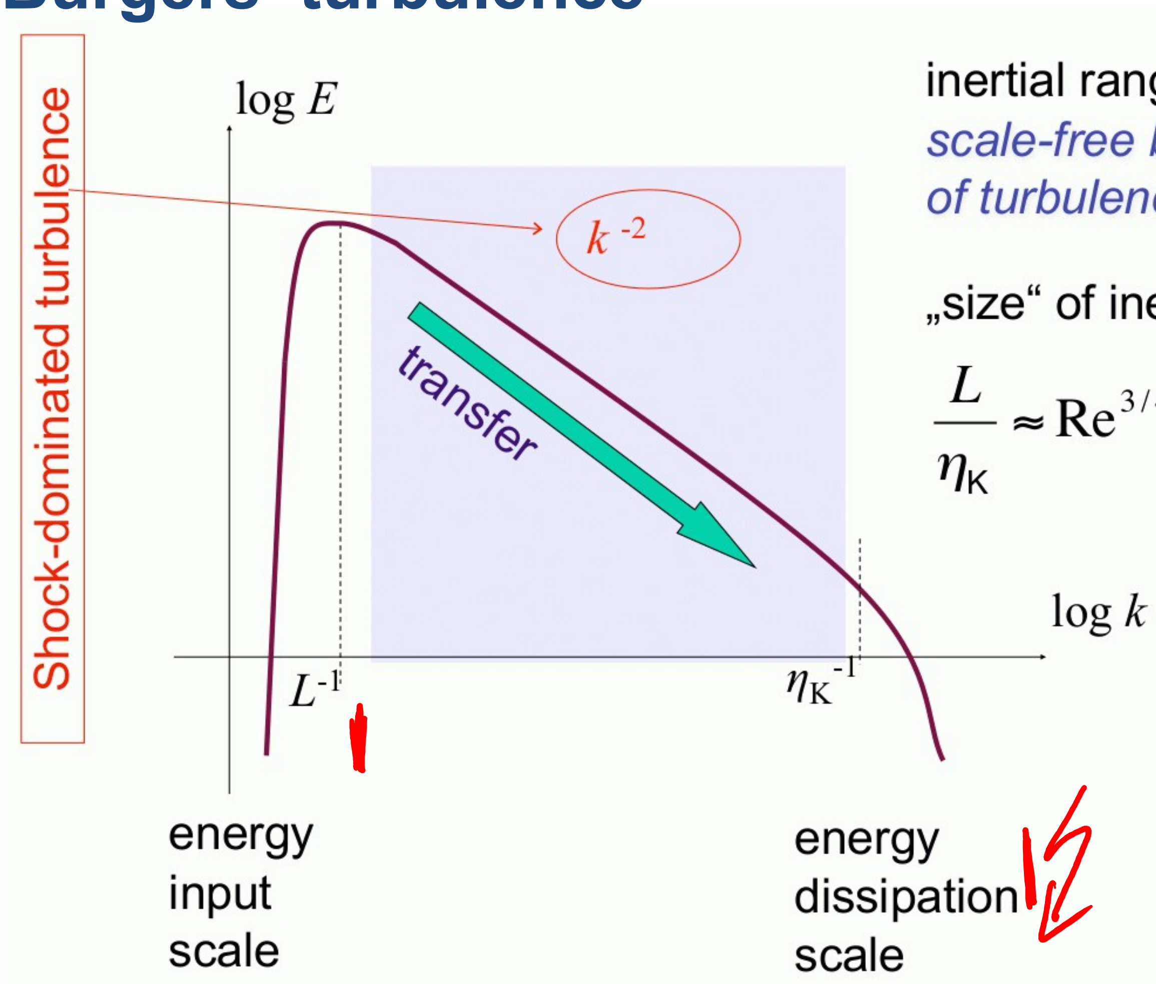
    $= \frac{1}{(2\pi)^{3/2}} \int \Theta(x) e^{i\vec{k}\vec{r}} d^3 r \sim k^{-4}$

  - **Energy spectrum**: $E(k) \sim k^{-2}$

  - Superposition of many shocks in many directions
    - Fourier transform is linear → No change: $E(k) \propto k^{-2}$
    - Burgers' turbulence

Turbulence

Burgers‘ turbulence



inertial range:
scale-free behavior of turbulence

„size“ of inertial range:

$$\frac{L}{\eta_K} \approx \mathrm{Re}^{3/4}$$

from R.Klessen (2012)

Turbulence observations

General results

- **Compressible vs. incompressible turbulence**

- Completely different physics, but only small change on observable spectrum:

  - Kolmogorov:

$$v(l) \sim l^{1/3} \qquad E(k) \sim k^{-5/3}$$

  - Burgers‘:

$$v(l) \sim l^{1/2} \qquad E(k) \sim k^{-2}$$

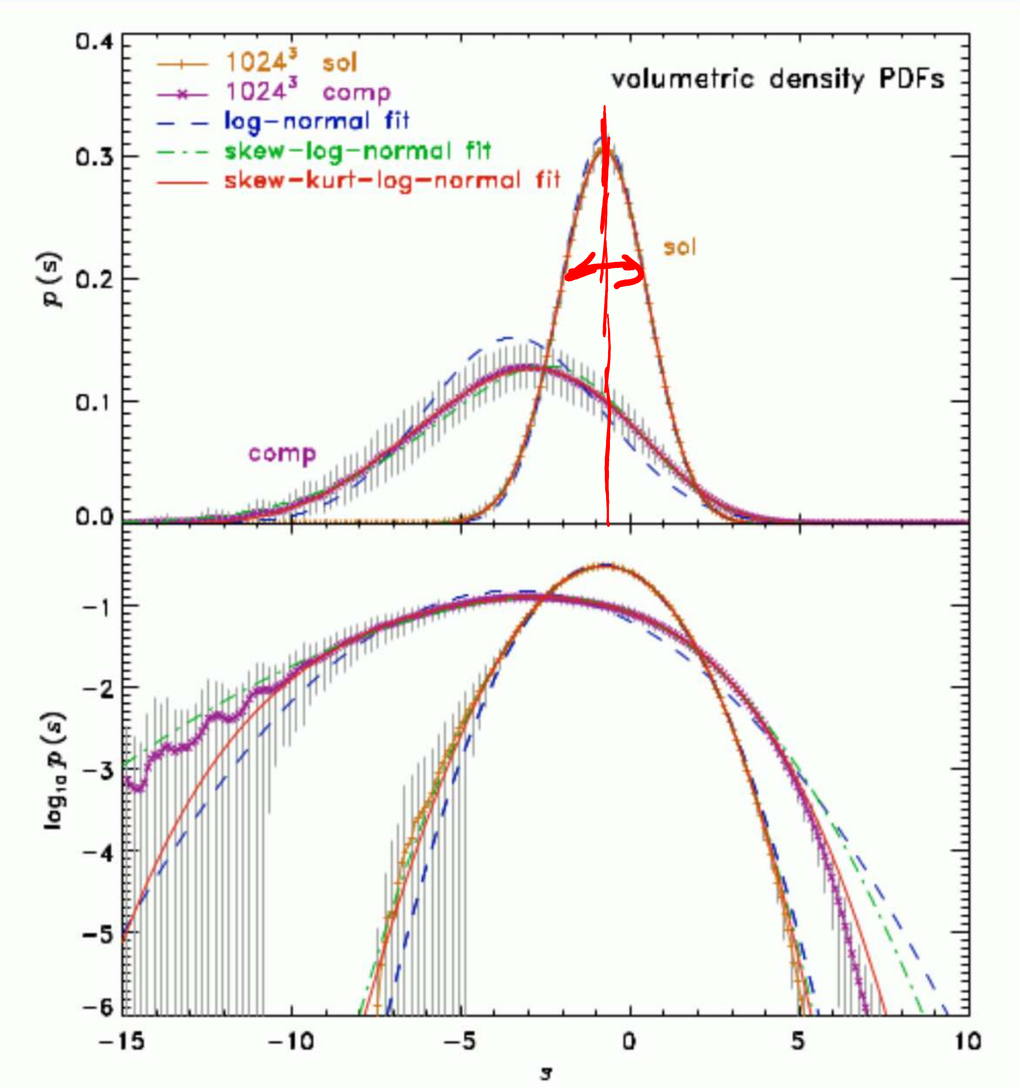
Turbulence observations

General results

- Velocity exponents 1/3….1/2
  - Kolmogorov: incompressible hydrodynamics
    $\rightarrow$ no density structure
  - Burger's turbulence: box full of shocks (step function)
    $\rightarrow$ full compressibility

- Actual density structure
  - Log-normal density distribution
    $$p(s) = \frac{1}{\sqrt{2\pi}\sigma_s} \exp\left(\frac{[s-s_0]^2}{2\sigma_s^2}\right)$$
  - with $s = \ln\left(\frac{n}{\langle n \rangle}\right)$

Federrath (2010)



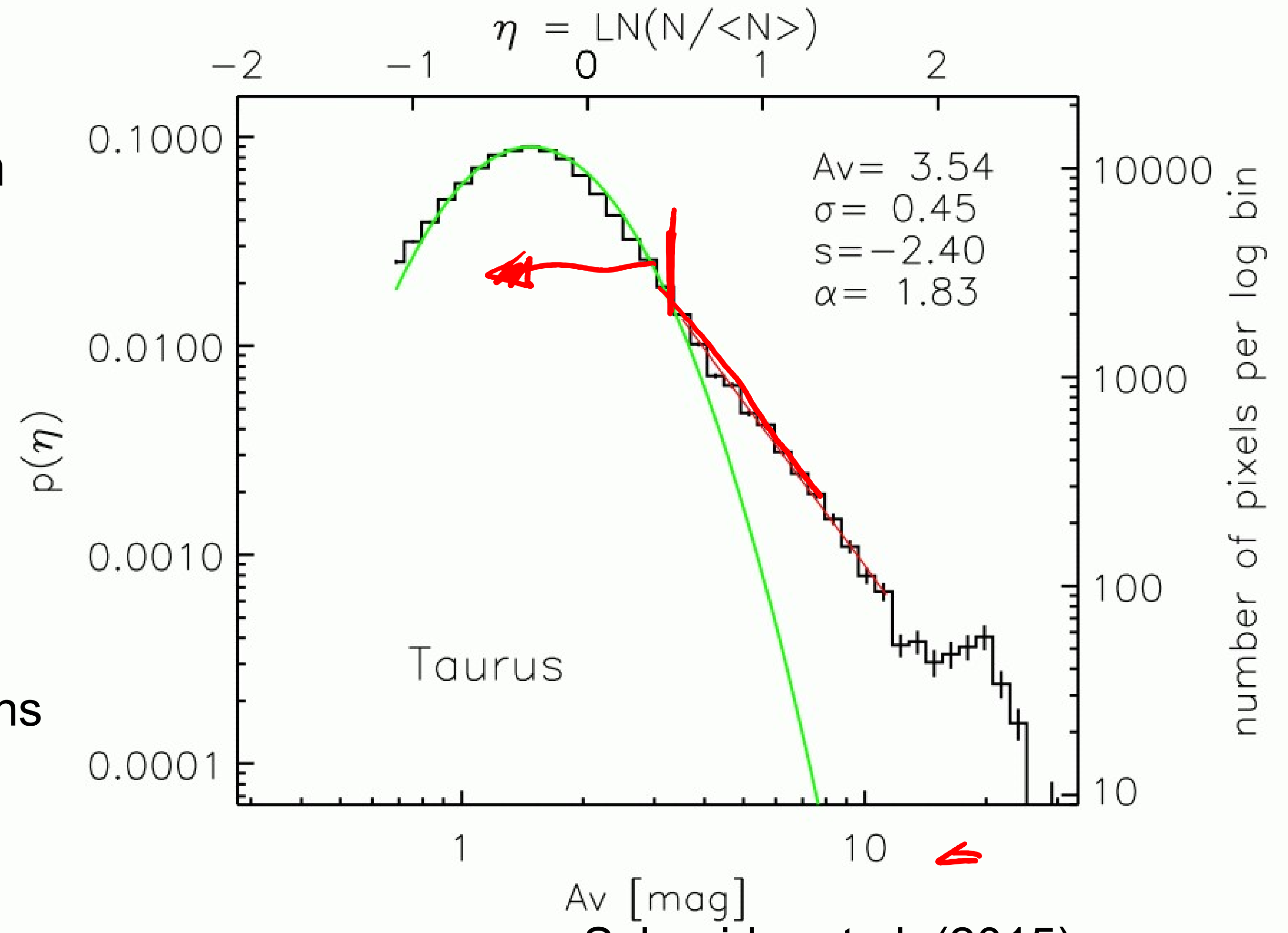
Turbulence observations

General results

- Log-normal density distribution
  - Width given by Mach number

$$\sigma_s^2 = \ln(1 + b^2\mathcal{M}^2)$$

    - b = parameter for the geometry of driving
  - Theoretical background still unclear
    - from numerical simulations
  - Observations confirm log-normal distribution
    - Additional tail from gravitational collapse



Schneider et al. (2015)

Turbulence observations

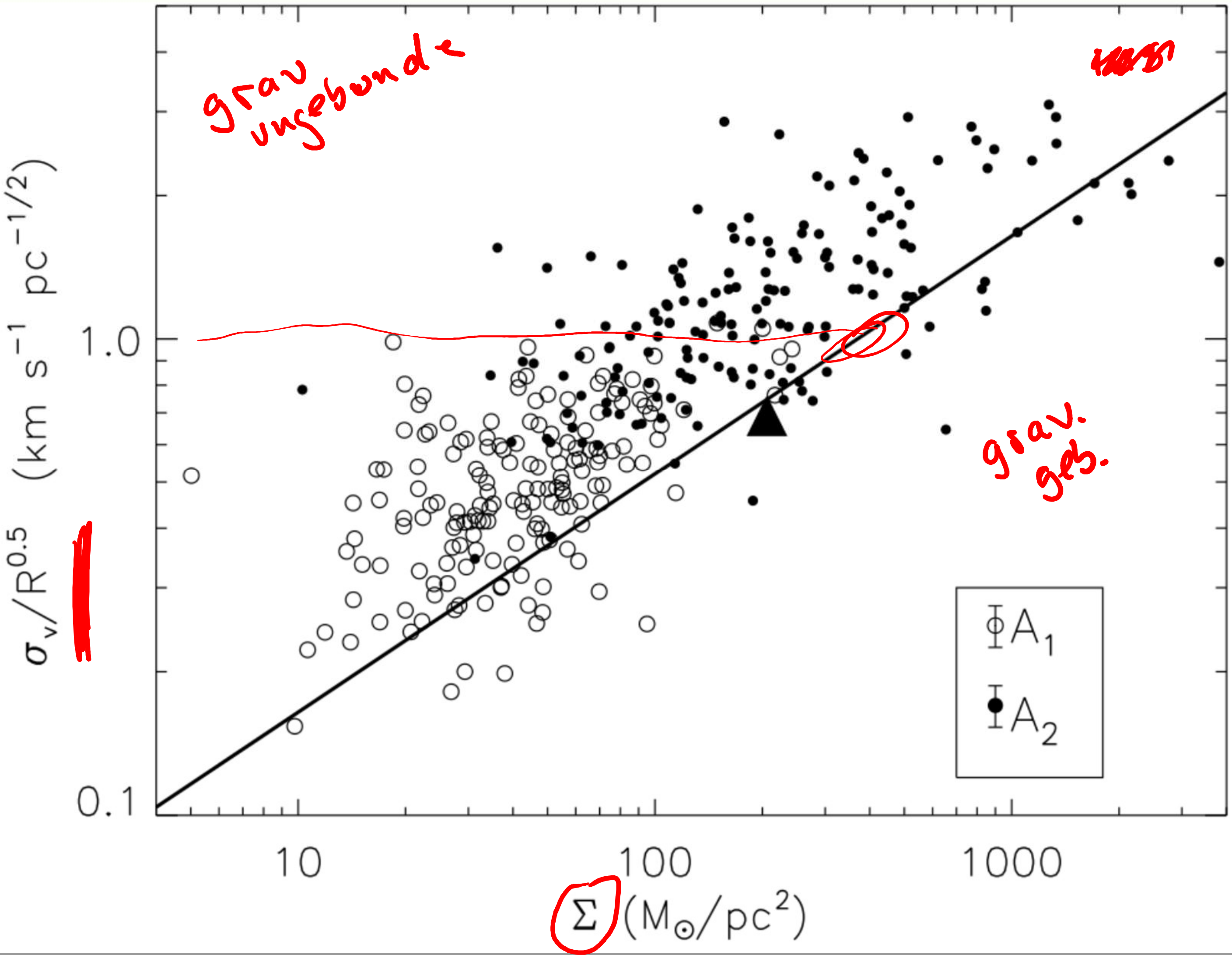
General results

- Density dependent velocity scaling relations (Heyer 2009)

- $\Delta v(l) \propto l^{0.5}$

  only for specific column density

$\sigma_v = V_0 \cdot R^{1/2} \cdot \Sigma^{1/2}$



Turbulence observations

Intermittency

- Gas in phase space:
  - Quiet regions
  - Missing velocities
  - Non-Gaussian statistics
- Confirmed by observations
  - **Non-Gaussian statistics** (Falgarone&Phillips 1990)

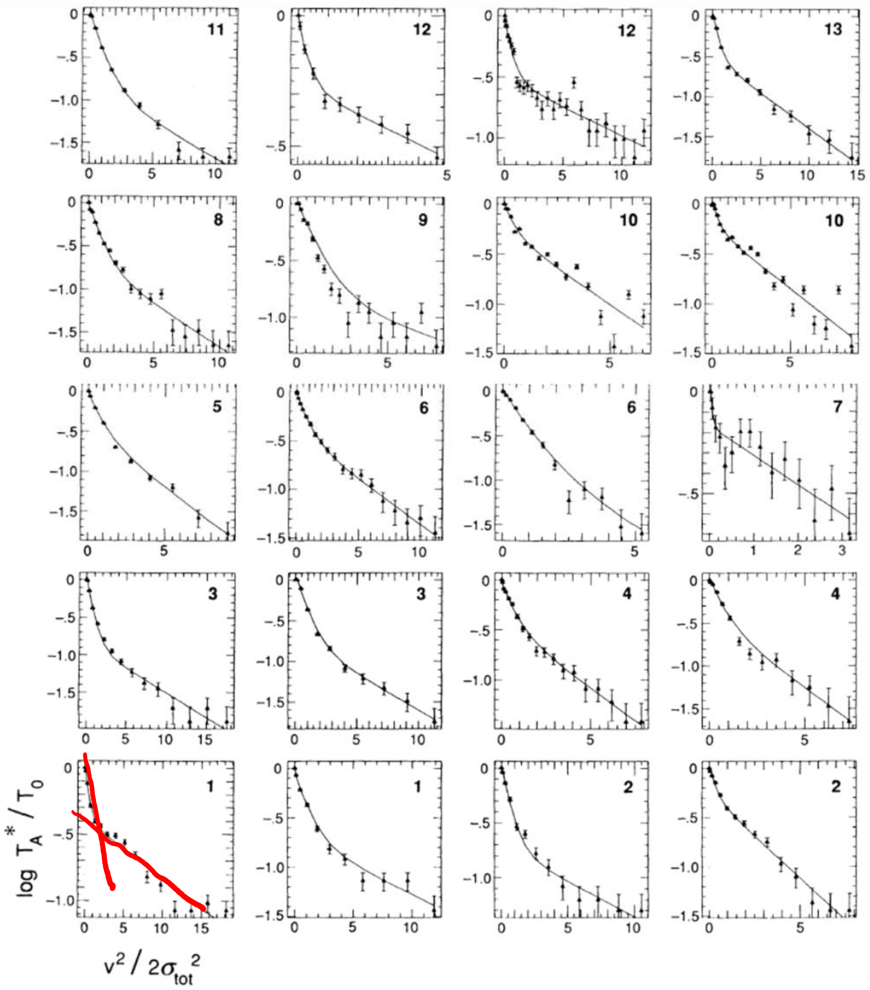
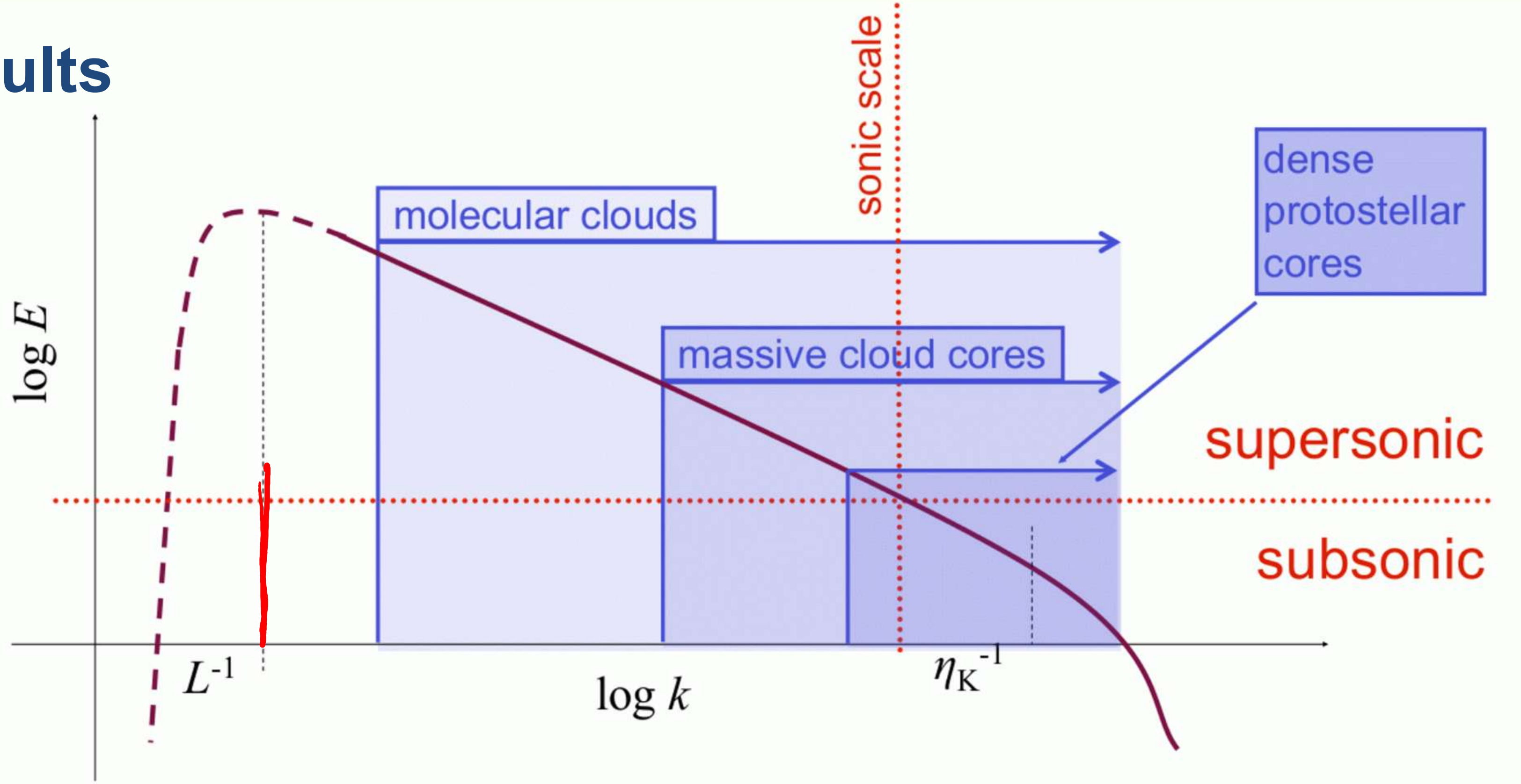


FIG. 2.—Linear-log display of 32 half-profiles, normalized to the peak temperature; ±1 σ error bars are shown. The continuous line represents the result of the linear fits through the wing and core channels, i.e., a two-Gaussian fit. Each profile is labeled by its entry number in Table 1.

Turbulence observations

General results



energy source & scale *NOT known* (supernovae, winds, spiral density waves?)

σ_{rms} << 1 km/s
$M_{rms} \leq 1$
$L \approx 0.1$ pc

dissipation scale not known (ambipolar diffusion, molecular diffusion?)

Turbulence observations

General results

micro turbulence

- Impact on ISM evolution
  - Additional turbulent pressure: $p(l) = \rho\left(c_s^3 + v_{turb}^2(l)\right)$
  - Mixing of regions and species
  - Transport of angular momentum
  - Curling of magnetic fields

QUIZ

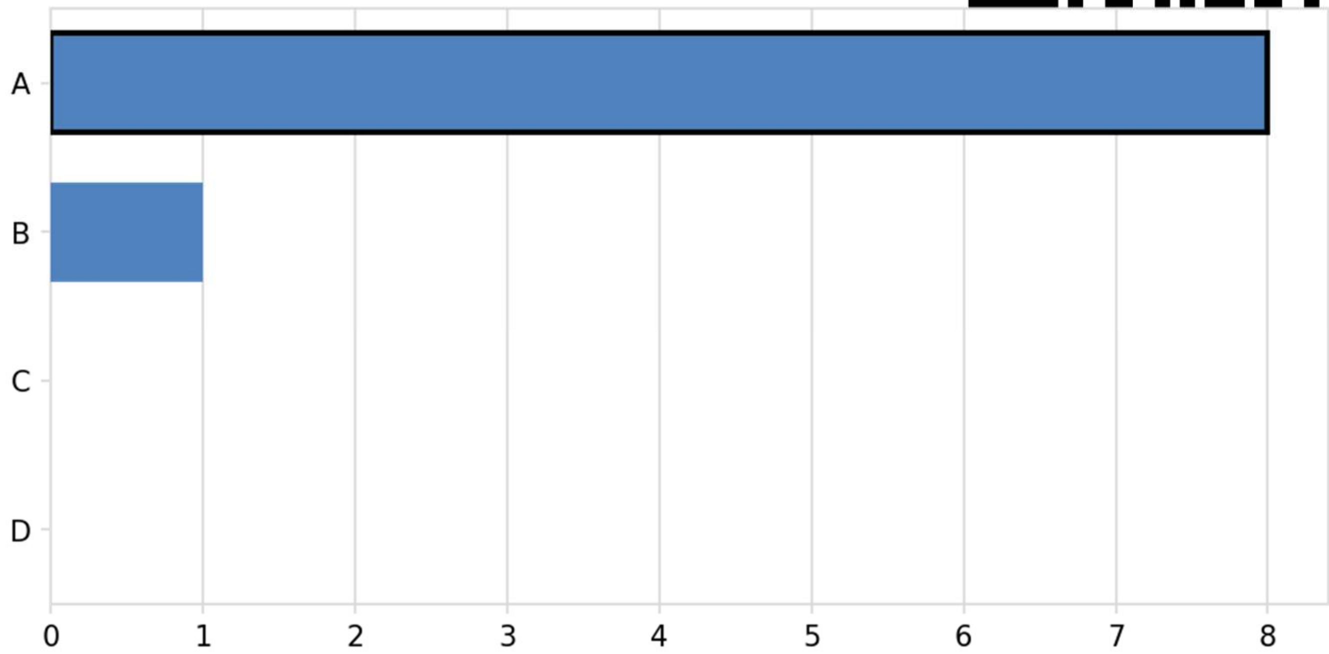
Wie lautet das Kolmogorov-Energiespektrum?

A) $E(k) \propto k^{-5/3}$

B) $E(k) \propto k^{-2}$

C) $E(k) \propto k^{-1}$

D) $E(k) \propto k^{-1\ /3}$



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9 Teilnehmer / Umfrage geschlossen

QUIZ