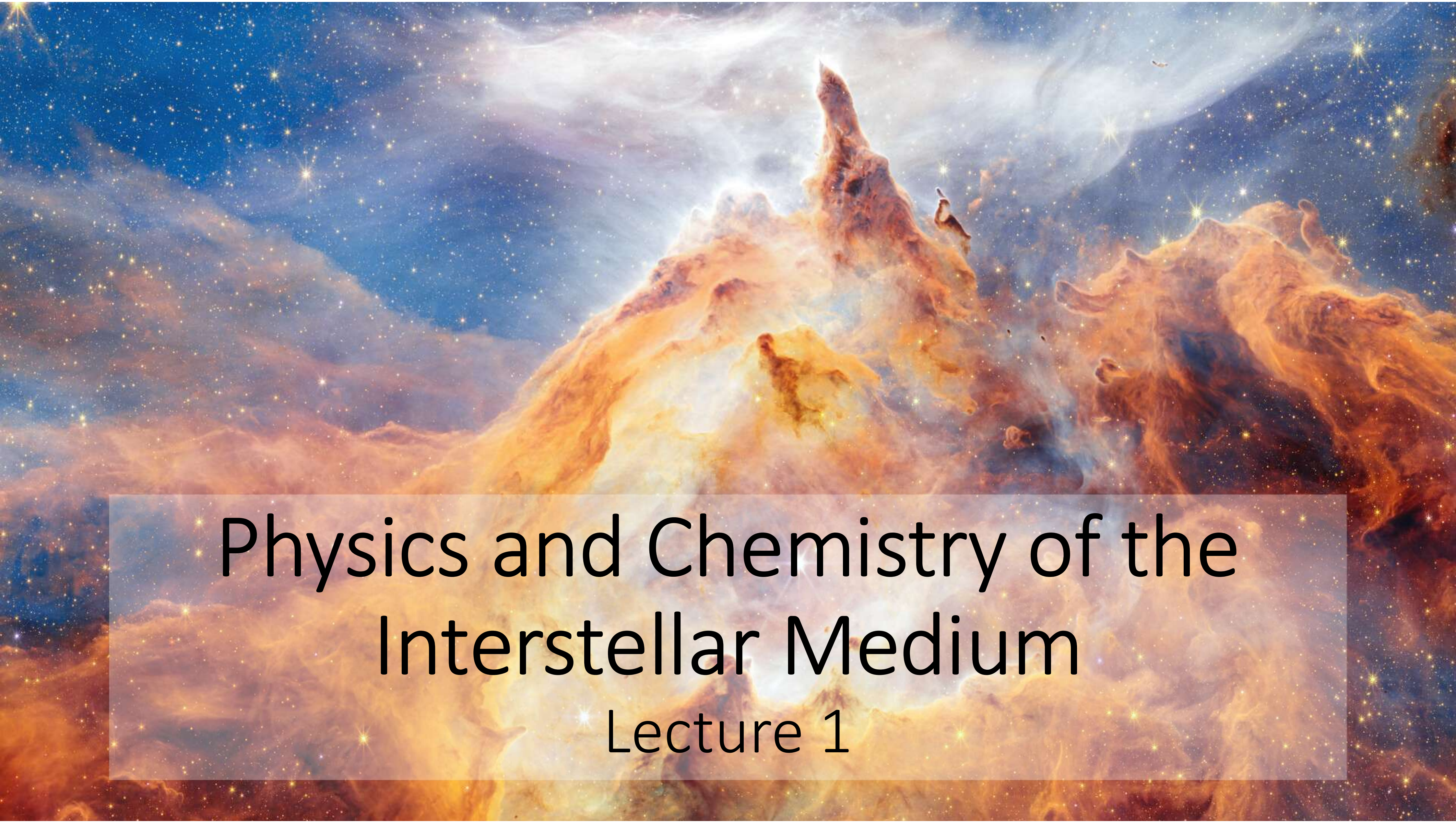


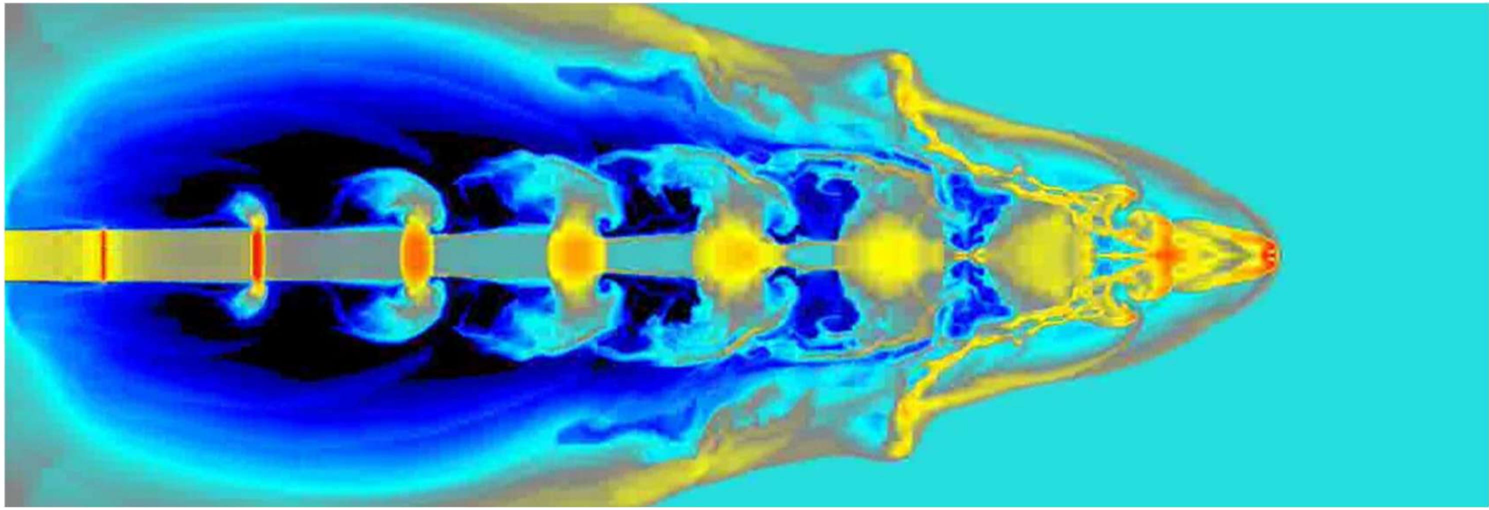
Physics and Chemistry of the Interstellar Medium

Lecture 1

The dynamics of interstellar gas

Lecture I

1. Hydrodynamics and Magnetohydrodynamics
   1.1. Fundamental equations



QUIZ

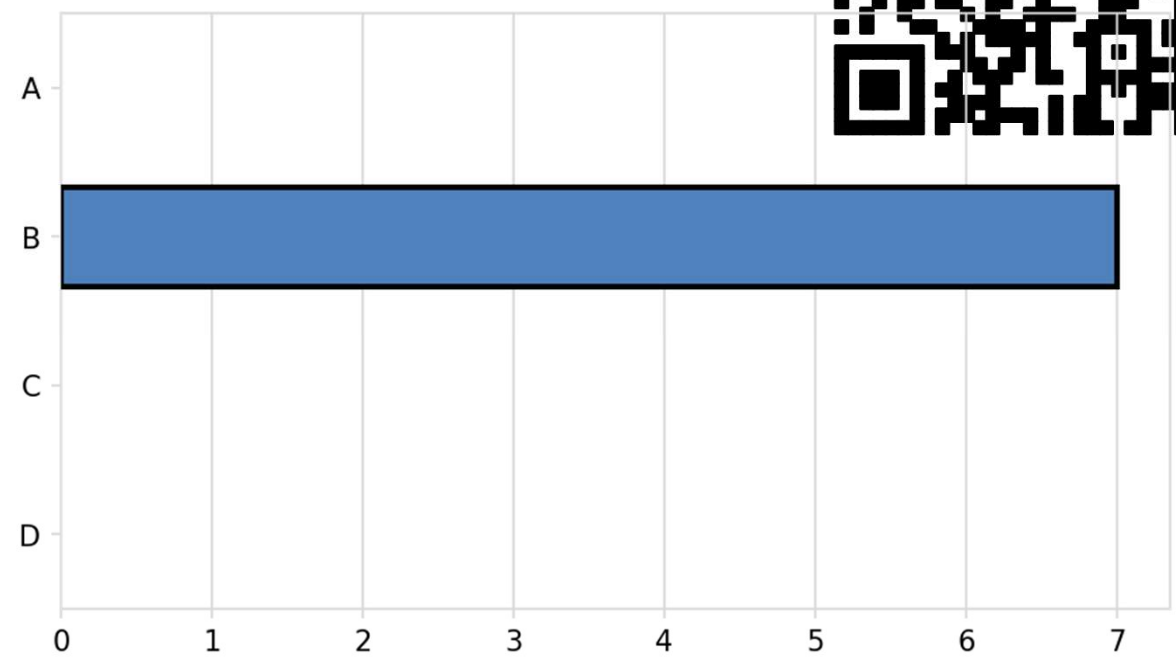
Was beschreibt der konvektive Term $\vec{v} \cdot \nabla \vec{v}$ in der totalen Ableitung?

A) Lokale zeitliche Beschleunigung

B) Advektion / konvektive Beschleunigung

C) Reibungskraft

D) Gravitationsbeschleunigung



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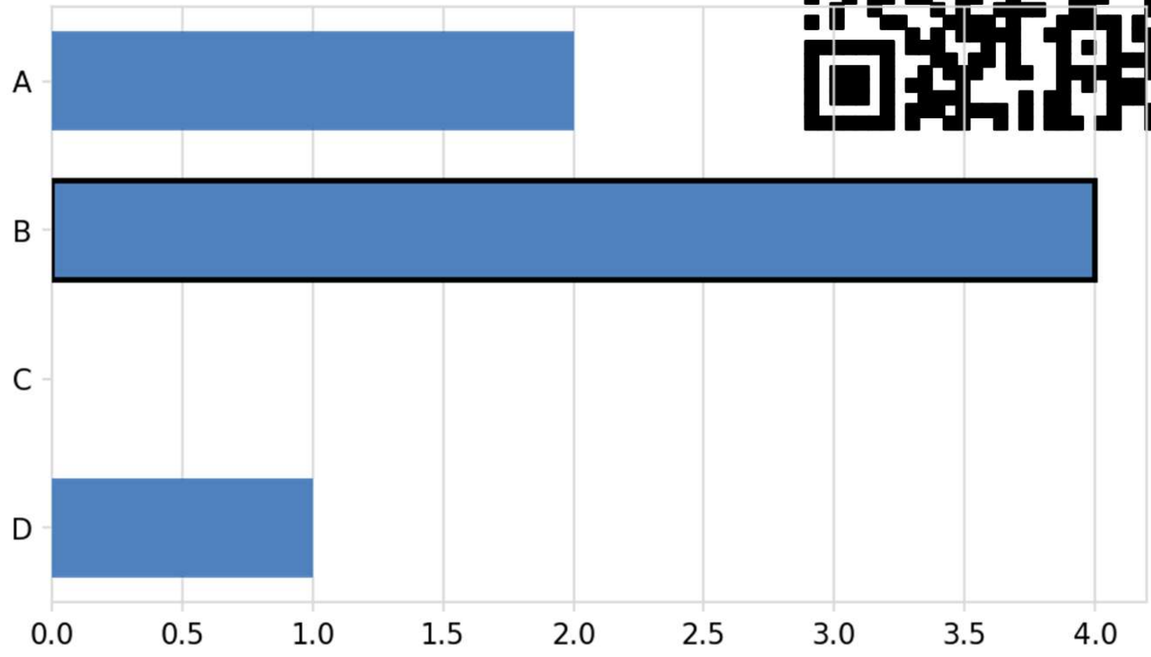
7 Teilnehmer / Umfrage geschlossen

QUIZ

In der Navier-Stokes-Gleichung taucht der Term $(\eta/3 + \zeta)\nabla(\nabla \cdot \vec{v})$ auf. Wann verschwindet dieser Term identisch?

A) Bei verschwindender Scherviskosität
B) Bei inkompressibler Strömung
C) Bei isothermer Strömung
D) Bei stationärer Strömung



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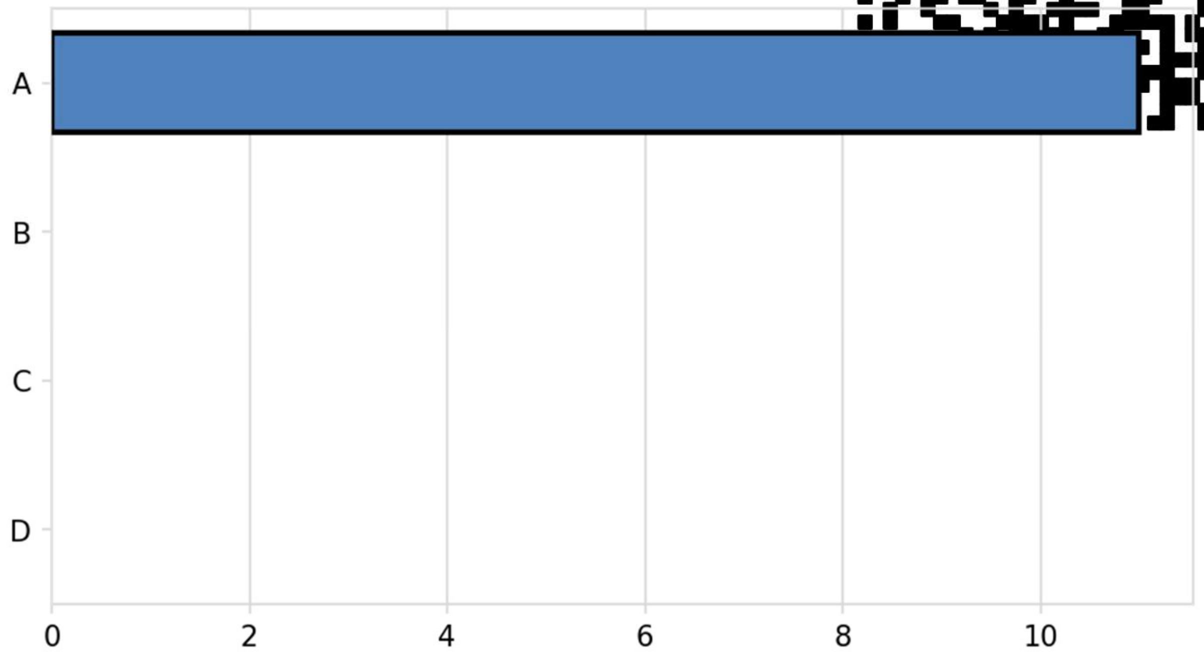
7 Teilnehmer / Umfrage geschlossen

QUIZ

Warum ist die polytrope Zustandsgleichung $p \propto \rho^{\gamma}$ für das ISM im Allgemeinen eine Überidealisierung?

A) Weil Energieaustausch mit der Umgebung die innere Energie ändert
B) Weil γ immer gleich 1 ist
C) Weil die Kontinuitäts- gleichung verletzt wird
D) Weil sie nur für monoatomare Gase gilt



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11 Teilnehmer / Umfrage geschlossen

The caloric equation of state

Ideal gas

- Use equation of state: $p = \frac{k_B}{\mu_{mol}} \rho T$
- To compute sound speed: $c_s^2 = \frac{\partial p}{\partial \rho} = \gamma \frac{p}{\rho}$ $\rightarrow c_s = \sqrt{\frac{\gamma k_B T}{\mu_{mol}}}$
  - Examples:
    - Earth atmosphere: N_2: $\gamma = \frac{7}{5}, \mu_{mol} = 4.7 \times 10^{-2}\ kg, T = 293\ K$
      $\rightarrow c_s = 350\ m/s$
    - Atomic ISM: HI +10% He: $\gamma = \frac{5}{3}, \mu_{mol} = 2.2 \times 10^{-27} kg, T = 100\ K$
      $\rightarrow c_s = 1\ km/s$
  - Notation: Use particle number density: $n = \frac{\rho}{\mu_{mol}} \rightarrow p = n\, k_B T$ $[cm^{-3}]$

But interstellar gas is not adiabatic! → **Heat sinks and sources**

*

Fundamental equations

The hydrodynamic equations

The heat transfer equation (balance for internal energy)

$$\rho \frac{\partial e}{\partial t} = -p\nabla\vec{v}$$

= mechanical work

$$+\kappa \Delta T$$

heat diffusion
κ = diffusion coeff.

$$+\frac{j^2}{\sigma}$$

=electric heating (currents)

$$+2\eta V_{ik}V_{ik} - \left(\frac{2}{3}\eta - \zeta\right)(\nabla\vec{v})^2$$

=frictional heating

V_{ik}=deformation tensor
uses Einstein notation:

$$V_{ik} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)$$

$$+\Gamma$$

=radiative heating

$$-\Lambda$$

=radiative cooling

Fundamental equation

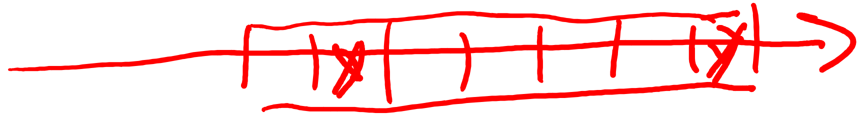
Summary

1. $\rho \frac{d\vec{v}}{dt} = \rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}\nabla)\vec{v} = -\nabla p - \rho \nabla U_{grav} + \eta \Delta \vec{v} + \left(\frac{\eta}{3} + \zeta\right) \nabla(\nabla \vec{v}) + \vec{j} \times \vec{B}$
2. $\Delta U_{grav} = 4\pi G \rho$
3. $\frac{\partial \rho}{\partial t} + \nabla(\rho \vec{v}) = 0$
4. $p = p(\rho, T)$ often simplified to: $\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$ otherwise:
5. $e = e(\rho, T)$
6. $\rho \frac{\partial e}{\partial t} = \kappa \Delta T - p \nabla \vec{v} + 2\eta V_{ik} V_{ik} - \left(\frac{2\eta}{3} - \zeta\right) (\nabla \vec{v})^2 + \frac{\vec{j}^2}{\sigma} + \Gamma - \Lambda$

Fundamental equations

Summary

- **Hydrodynamic equations = Navier-Stokes + Poisson + continuity + equation of state + caloric equation of state + heat transfer equation**
  - Includes all gas flows and energy balance
    - Needs to be considered individually for each species!
  - Not closed due to
    - Electromagnetic terms
    - Radiation terms
    - Gas-dust interaction



- Fundamental flaw: Ignores chemical reactions $\frac{dn_i}{dt} = \sum_j k_{ij} n_j n_k - \sum_l k_{li} n_l n_i + \cdots$
  - Continuity equation not fulfilled for individual species (only for overall mass) $H_2 \leftrightarrow H + H$
  - Chemical heating/cooling in heat transfer equation missing

The hydrodynamic equations

Applications

- Most simple, non-trivial example: **Self-gravitating hydrostatic sphere**
  - 1-D (spherical symmetry)
  - Mass contained in radius r: $\frac{dM}{dr} = 4\pi r^2 \rho(r)$

    Integrate: $M(r) = \int_0^r 4\pi r^2 \rho(r)\, dr$
  - Poisson equation: $\Delta U_{grav} = 4\pi G \rho$

    In spherical symmetry: $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial U_{grav}}{\partial r}\right) = 4\pi G \rho(r)$
  - Integration using known mass integral:

$$r^2 \frac{\partial U_{grav}}{\partial r} = G \int_0^r 4\pi \rho(r') r'^2 dr' = G M(r)$$

The hydrodynamic equations

Self-gravitating hydrostatic sphere

- Hydrostatic, i.e. balance between thermal pressure and gravity
  - $\nabla p = -\rho \nabla U_{grav}$
  - **In spherical symmetry**: $\frac{\partial p}{\partial r} = -\rho \frac{\partial U_{g}}{\partial r}$
  - With results from Poisson integration: $r^2 \frac{\partial U_{grav}}{\partial r} = GM(r)$

$\frac{\partial p}{\partial r} = -\rho \frac{GM(r)}{r^2}$ = general balance for hydrostatic sphere

  - With polytropic equation of state $c_s^2 = \frac{\partial p}{\partial \rho}$

$\frac{\partial p}{\partial r} = \frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial r}$ $\quad c_s^2 \frac{1}{\rho}\frac{\partial \rho}{\partial r} = -\frac{GM(r)}{r^2}$ → differential equation for ρ = closed system

The hydrodynamaic equations

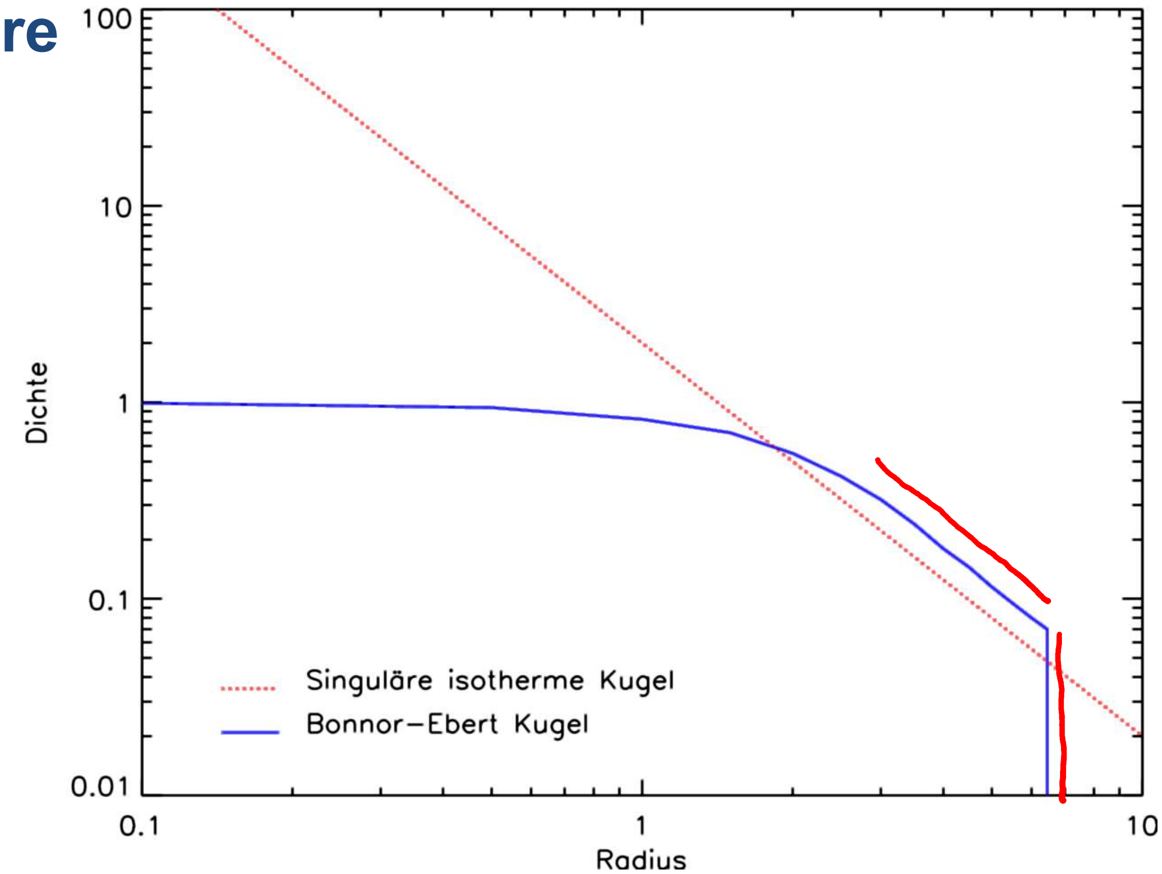
Self-gravitating hydrostatic sphere

- differential equation $c_s^2 \frac{1}{\rho}\frac{\partial \rho}{\partial r} = -\frac{GM(r)}{r^2}$
  - Solution by power-law ansatz: $\rho(r) = \rho_0 r^{-\alpha}$
  - Mass-integral $M(r) = 4\pi \int_0^r \rho_0 r'^{2-\alpha} dr' = \frac{4\pi}{3-\alpha} r^{3-\alpha}$
  - Pressure term: $\frac{1}{\rho}\frac{\partial \rho}{\partial r} = -\alpha \frac{1}{r}$
  - Together: $c_s^2 \left(-\frac{\alpha}{r}\right) = -G \frac{4\pi}{3-\alpha} \rho_0 r^{1-\alpha}$
    - Exponent from radial dependence: $\alpha = 2$
    - Prefactor: $\rho_0 = \frac{c_s^2}{2\pi G}$ => $\rho(r) = \frac{c_s^2}{2\pi G r^2}$ =singular isothermal sphere

The hydrodynamic equations

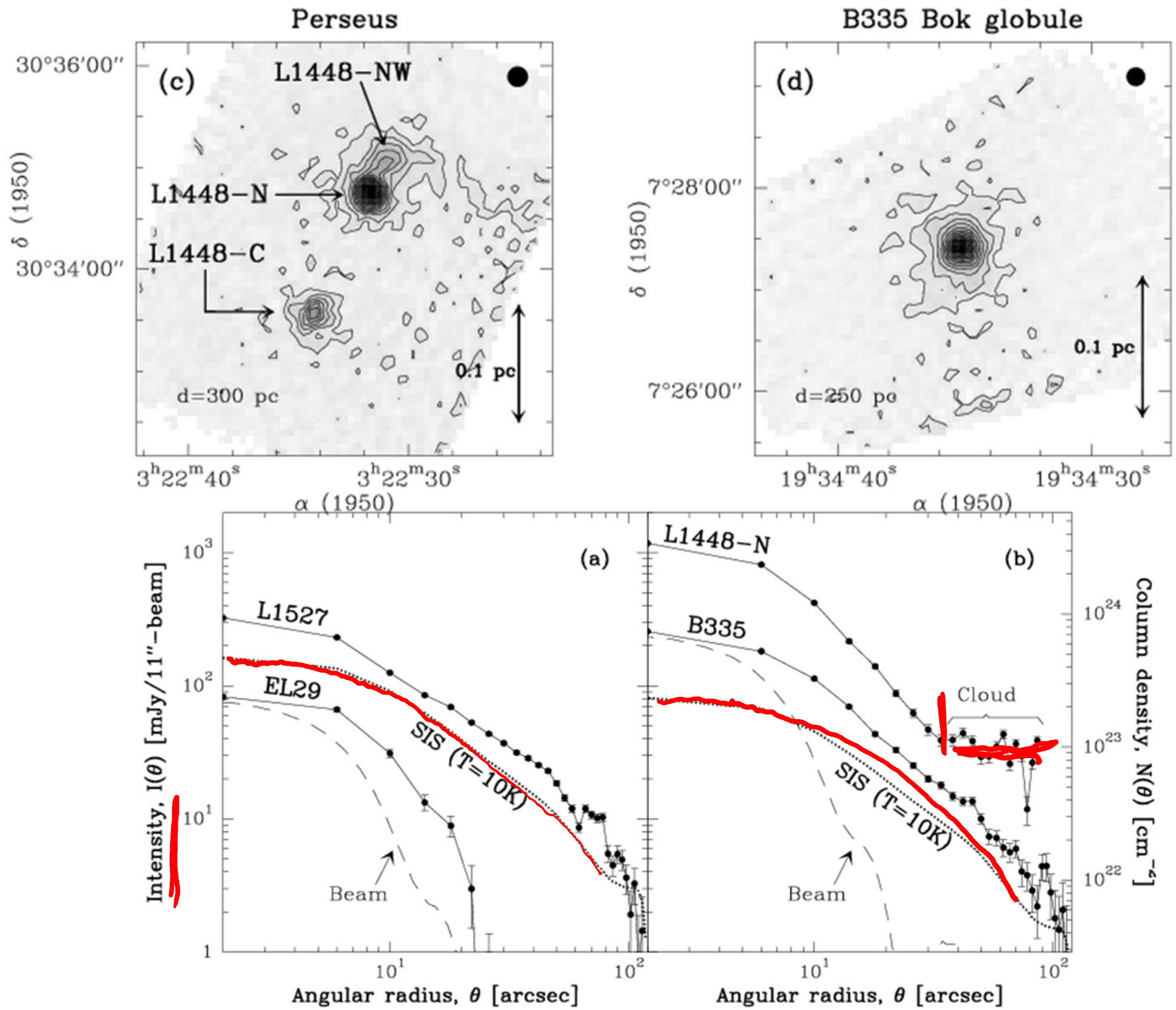
Self-gravitating hydrostatic sphere

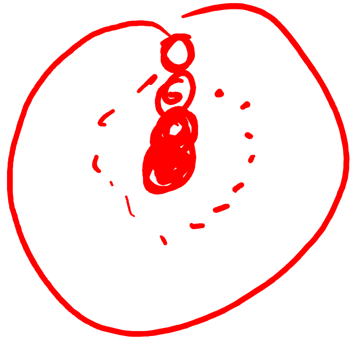
- Alternative solution:
  - Avoid singularity in center
  - Assume hard external boundary with given pressure instead
  - **Bonnor-Ebert-sphere**



Comparison of singular sphere with Bonnor-Ebert profile

Observations





Motte & Andre (1999)

Relative Estimate of effects

Navier-Stokes equation

- Ignore volume viscosity ζ and Lorentz force:

$$\frac{\rho\partial\vec{v}}{\partial t} + \rho(\vec{v}\nabla)\vec{r} = -\nabla\rho - \rho\nabla U_{grav} + \eta\Delta\vec{v}$$

  - assume stationary flow: $\rho\frac{\partial\vec{v}}{\partial r} = 0$
  - with sound speed: $c_s^2 = \frac{dp}{d\rho} \rightarrow \frac{\partial p}{\partial r} = c_s^2\frac{\partial\rho}{\partial r} \rightarrow \nabla p = c_s^2\nabla\rho$
  - and Poisson equation: $\nabla U_{grav} = \frac{GM(r)}{r^2}$ $\rightarrow |\nabla U_g| \sim 4\pi G \rho L$

L: char. Länge

V: char. Geschw.

- Estimate through gradients over characteristic scale L

$$\rho\frac{V^2}{L} = -\rho\frac{c_s^2}{L} - \rho 4\pi G\rho L + \eta\frac{V}{L^2}$$

Relative estimate of effects

Characteristic numbers

- Forces in Navier-Stokes: $$\rho \frac{v^2}{L} = -\rho \frac{c_s^2}{L} - \rho \times 4\pi G \rho L + \eta \frac{v}{L^2}$$

streaming motion gas pressure gravity viscosity

- Compare pressure and streaming motion: $v^2 \leftrightarrow c_s^2$
  - Characteristic number: $\mathcal{M} = \frac{v}{c_s}$ **Mach number**

  If $\mathcal{M} \gg 1$ thermal pressure is not relevant for the gas flow.

- Example for cold atomic ISM:
  - $c_s \approx 1 \frac{km}{s}$ $v = 10 \frac{km}{s}$ $\mathcal{M} \approx 10$
  - $\mathcal{M} = 10$ Thermal pressure is not important

Relative estimate of effects

Characteristic numbers

- Forces in Navier-Stokes: $$\rho \frac{v^2}{L} = -\rho \frac{c_s^2}{L} - \rho \times 4\pi G \rho L + \eta \frac{v}{L^2}$$
  streaming motion gas pressure gravity viscosity
- Compare viscosity and streaming motion: $\rho v \leftrightarrow \frac{\eta}{L} = \rho \frac{\nu}{L}$
  with kinematic viscosity $\nu = \eta / \rho$
  - Characteristic number: $R_e = \frac{vL}{\nu}$ **Reynolds number**

    If $R_e \gg 1$ viscous dissipation is not relevant for the gas flow.
- Example for cold atomic ISM:
  - $L \approx 1\,\mathrm{pc} = 3.08 \cdot 10^{18}\,\mathrm{cm}$
  - Viscosity ν determined by thermal collisions in the gas

Relative Estimate of effects

Reynolds number

- Viscosity ν determined by thermal collisions in the gas $\nu = \frac{c_s \lambda}{3}$

with $\lambda = \frac{1}{n\sigma_{coll}}$ = mean free path length in the gas

- HI gas: $\sigma_H \approx 1 \times 10^{-16} cm^2$
- Combine $Re = \frac{3 v L}{c_s} \sigma_{coll} \cdot n$

$$= 3 \frac{10 km/s}{1 km/s} 3 \times 10^{18} cm \times 10^{-16} cm^2 \times 30 cm^{-3}$$

$$= 27.000 \gg 1$$

viscous dissipation is negligible for the structure of the gas flow

- Medium is highly turbulent (next lecture)

Relative estimate of effects

Characteristic numbers

- Forces in Navier-Stokes: $\rho \frac{v^2}{L} = -\rho \frac{c_S^2}{L} - \rho \times 4\pi G \rho L + \eta \frac{v}{L^2}$

  streaming motion gas pressure gravity viscosity

- Compare gravity and streaming motion $\frac{v^2}{L^2} \leftrightarrow 4\pi G \rho$
  - Characteristic number: $\alpha = \frac{v^2}{G\rho L^2}$ **Virial parameter**

    If $\alpha \gg 1$ gravity is not relevant for the gas flow.

- Example for cold atomic ISM:

  $\alpha = \frac{10^8 m^2}{s^2} / (6.7 \times \frac{10^{-11} m^3}{kg\, s^2}\, 1.6 \times 10^{-27} kg\; 3 \times 10^7 m^{-3}\; 10^{33} m^2)$

  $\alpha = 30.000$ Gravity is not important

Characteristic numbers

- Forces in Navier-Stokes: $\rho \frac{v^2}{L} = -\rho \frac{c_s^2}{L} - \rho \times 4\pi G \rho L + \eta \frac{v}{L^2}$

  streaming motion gas pressure gravity viscosity

- Compare gravity and thermal pressure: $\frac{c_s^2}{L^2} \Leftrightarrow 4\pi G\rho$

  - Characteristic number: $L_J = \sqrt{\frac{c_s^2}{G\rho}}$ **Jeans Length**

  If $L_J \gg 1$ gravity is not important relative to the thermal pressure.

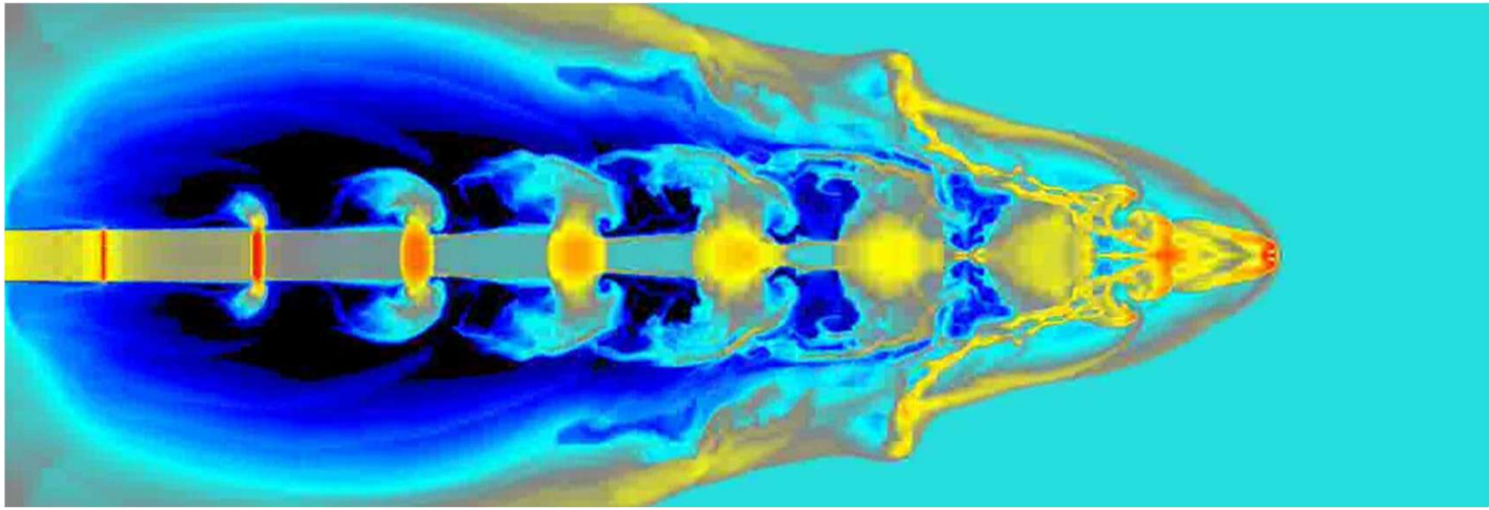
- Example for cold atomic ISM:
  - See later this lecture
  - $L_J = 18000$ Gravity is not important

The dynamics of interstellar gas

Lecture II

1. Hydrodynamics and Magnetohydrodynamics

1.2 Magnetohydrodynamics



2. Instabilities

2.1 Jeans Instability

Review - Hydrodynamics

1. Fundamental equation

$$\frac{\rho\partial\vec{v}}{\partial t} + \rho(\vec{v}\nabla)\vec{v} = -\nabla p - \rho\nabla U_{grav} + \eta\Delta\vec{v} + \left(\frac{\eta}{3} + \zeta\right)\nabla(\nabla\vec{v}) + \vec{j}\times\vec{B}$$

2. $\Delta U_{grav} = 4\pi G\rho$
3. $\frac{\partial\rho}{\partial t} + \nabla(\rho\vec{v}) = 0$
4. $p = p(\rho, T)$ often simplified to: $\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$ otherwise:
5. $e = e(\rho, T)$
6. $\rho\frac{\partial e}{\partial t} = \kappa\Delta T - p\nabla\vec{v} + 2\eta V_{ik}V_{ik} - \left(\frac{2\eta}{3} - \zeta\right)(\nabla\vec{v})^2 + \frac{\vec{j}^2}{\sigma} + \Gamma - \Lambda$

Magnetohydrodynamics

Motivation

- **Missing terms in hydrodynamic equations**
  - Equation of motion: $\vec{j} \times \vec{B}$
  - Energy Balance: j^2/σ

Current density: $\vec{j} = qX_i\vec{v}n$
X_i = degree of ionization
q = particle charge

- **Main problem:** magnetic field $\vec{B}$ needed
  - can be computed through induction equation:

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \frac{1}{\mu\sigma} \Delta \vec{B}$$

μ = magnetic permeability

Computation of the magnetic fields

- **Problem:** Induction equation assumes that $\mu,\ \sigma$ are constants
  - In a highly compressible gas this is not the case
    - Density and ionization degree change with $\vec{r}$
    - In a moving fluid this also changes in time t with the motion of the gas

- **Solution:** Co-moving reference frame of fluid dynamics
  - Moves with every „cell“ of gas
  - There $\mu, \sigma, \eta, \zeta, \kappa$ can be assumed to be constant
  - Needs induction equation in co-moving frame
    - Electrodynamics in co-moving frame

- **Magnetohydrodynamics (MHD)**

Electrodynamics from scratch

- **Maxwell equations in inertial system Σ**
  - $\nabla \vec{B} = 0$ Magnetic field
  - $\nabla \times \vec{E} = -\dot{\vec{B}}$ Electric field
  - $\nabla \vec{D} = \rho_{el}$ Electric displacement field
  - $\nabla \times H = \vec{j} + \dot{\vec{D}}$ magnetic field strength

- But in **co-moving system Σ'** we can write the material equations (M1-3)
  - $\vec{B}' = \mu \vec{H}'$ μ = permeability
  - $\vec{D}' = \epsilon \vec{E}'$ ϵ = permittivity/dielectric constant
  - $\vec{j}' = \sigma \vec{E}'$ σ = conductivity (Ohm‘s law)

  - They do not hold in the rest system!

Magnetohydrodynamics

Electrodynamics from scratch

- **Transformation needed:** $\Sigma \rightarrow \Sigma'$
  - **Problem:** Maxwell equations – invariant for Lorentz transformation
    Hydrodynamic equations – invariant for Galilei transformation
  - Does not fit together
  - Can only be unified in approximation: $|\vec{v}| \ll c$

„Classical MHD“

  - Then use Galilei transformation: $\vec{r}' = \vec{r} - \vec{v} \cdot t$, $t' = t$
    - In co-moving frame Σ': $\vec{v}'(\vec{r}'_0, t'_0) = 0$
      - The medium is at rest at a given time t'_0 and location $\vec{r}_0$

Magnetohydrodynamics

Galilei transform: $\Sigma \rightarrow \Sigma'$

- **Transfrom field equations**
  (neglecting relativistic terms)
- Approach:
  - Consider Lorentz force on particle with charge q in moving frame:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

  - q is at rest in Σ'. Force there can only be excerted by electric field $\vec{E}'$
  - Gives transformation for electric field:

$$\vec{E}' = \vec{E} + \vec{v} \times \vec{B}$$

Magnetohydrodynamics

Gallilei transform: $\Sigma \rightarrow \Sigma'$

- **Equivalent for all field equations**

$\vec{E}' = \vec{E} + \vec{v} \times \vec{B}$ $\vec{D}' = \vec{D} + \frac{1}{c^2}\vec{v} \times \vec{H}$

$\vec{H}' = \vec{H} - \vec{v} \times \vec{D}$ $\vec{B}' = \vec{B} - \frac{1}{c^2}\vec{v} \times \vec{E}$

$\vec{j}' = j - \rho_{el}\vec{v}$

- Factors $\frac{1}{c^2}$ stem from $\vec{H} \times \vec{E} = \frac{1}{\mu_0\epsilon_0}\vec{B} \times \vec{D} = c^2(\vec{B} \times \vec{D})$
- Current: $\vec{j}' = \vec{j} - \rho_{el}\vec{v}$ Alternatively: $\rho_{el}' = \rho_{el} - \frac{1}{c^2}\vec{v}/\vec{j}$

Currents compensate charge

Magnetohydrodynamics

Gallilei transform: $\Sigma \rightarrow \Sigma'$

- **Rewrite material equations (M1-3) for system at rest Σ**
  - Uses all of the equations above

$$\vec{B} - \frac{1}{c^2}(\vec{v} \times \vec{E}) = \mu(\vec{H} - \vec{v} \times \vec{D})$$

$$\vec{D} - \frac{1}{c^2}(\vec{v} \times \vec{H}) = \epsilon(\vec{E} - \vec{v} \times \vec{B})$$

$$\vec{j} - \rho_{el}\vec{v} = \sigma(\vec{E} - \vec{v} \times \vec{B})$$

  - Now combine with Maxwell equations for closed system:

$$\nabla\vec{B} = 0 \qquad \nabla \times \vec{E} = -\dot{\vec{B}} \qquad \nabla\vec{D} = \rho_{el} \qquad \nabla \times \vec{H} = \vec{j} + \dot{\vec{D}}$$

    - Quite complex overall system of 7 differential equations
    - Simplifications for handy solutions needed (stronger than $|\vec{v}| \ll c$)
    - MHD approximation

Magnetohydrodynamics

MHD approximations

- **Based on characteristics numbers:**
  - Induction equations: $\frac{\partial \vec{B}}{\partial t} = \nabla \times \left(\vec{v} \times \vec{B}\right) + \frac{1}{\mu\sigma}\Delta\vec{B}$
- Same dimensional estimates as for HD Navier-Stokes equation
  - assume stationary flow: $\frac{\partial \vec{B}}{\partial t} = 0$
  - Estimate through gradients over characteristic scale: L
  - Compare source (diffusion) term and curl motion:
  - Characteristic number: $R_m = v\,L\,\mu\,\sigma$ Magnetic Reynolds number
  - If $R_m \gg 1$ magnetic dissipation is not relevant → perfect conductivity
- Example for cold atomic ISM: $\mu\sigma = 10^{-22}\,\frac{s}{cm^2} \rightarrow R_m = 300$

Magnetohydrodynamics

MHD approximations

1. $R_m \gg 1$ **High conductivity over chracteristics length of system L**

   $\rightarrow \rho_{el} \times \vec{v} \ll \vec{j}$

   term can be dropped

2. **Characteristic velocity is small:**

   $\frac{c}{\sigma\mu L} \ll \frac{c}{\sqrt{\epsilon\mu}}$

   $\rightarrow \vec{v} \times \vec{D} \ll \vec{H} \quad \vec{v} \times \vec{E} \ll c^2 B$

   Displacement currents do not change the magnetic field

3. **Slow variation over characteristic timescale of the system τ:** $\frac{L}{\tau} \ll \frac{c}{\sqrt{\epsilon\mu}}$

   $\rightarrow \dot{D} \ll j$

   Induced polarization dropped

Limitations on velocity, size, and time scale

MHD approximations

- **Impact:**
  - Material equations

$$\vec{B} - \frac{1}{c^2}\left(\vec{v} \times \vec{E}\right) = \mu\left(\vec{H} - \vec{v} \times \vec{D}\right)$$

$$\vec{D} - \frac{1}{c^2}\left(\vec{v} \times \vec{H}\right) = \epsilon\left(\vec{E} - \vec{v} \times \vec{B}\right)$$

$$\vec{j} - \rho_{el}\vec{v} = \sigma\left(\vec{E} - \vec{v} \times \vec{B}\right)$$

  - Now combine with Maxwell equations for closed system:

$$\nabla\vec{B} = 0$$

$$\nabla \times \vec{E} = -\dot{\vec{B}}$$

$$\nabla\vec{D} = \rho_{el}$$

$$\nabla \times \vec{H} = \vec{j} + \dot{\vec{D}}$$

Major simplification

Equations for $\vec{D}$ no longer needed

MHD approximations

- **Result:**
  - 5 equations for MHD electrodynamics:

$$\nabla \vec{B} = 0$$

$$\nabla \times \vec{E} = -\dot{\vec{B}}$$

$$\nabla \times \vec{H} = \vec{j}$$

$$\vec{B} = \mu \vec{H}$$

$$\vec{j} = \sigma(\vec{E} - \vec{v} \times B)$$

Ohmic law in MHD flow

  - For 4 quantities: $\vec{B}, \vec{E}, \vec{H}, \vec{j}$
  - $\vec{j}$ is a direct result of the closed system above – can be eliminated
    - **4 fundamental equations of electrodynamics**

The induction equation

- **Maxwell equations with approximate material equations:**

$$\nabla\times\vec{E} = -\dot{\vec{B}}, \qquad \nabla\vec{B} = 0, \qquad \vec{B} = \mu\vec{H}$$
$$\nabla\times\vec{H} = \vec{j}, \qquad \vec{j} = \sigma(\vec{E} + \vec{v}\times\vec{B})$$

- Combine

$$-\frac{\partial\vec{B}}{\partial t} = \nabla\times\left(\frac{\vec{j}}{\sigma} - \vec{v}\times\vec{B}\right) = \nabla\times\left(\frac{\nabla\times\vec{B}}{\mu\sigma} - \vec{v}\times\vec{B}\right)$$

- With $\Delta\vec{A} = \nabla\left(\nabla\vec{A}\right) - \nabla\times\left(\nabla\times\vec{A}\right)$

$$\frac{\partial\vec{B}}{\partial t} = \frac{\Delta\vec{B}}{\mu\sigma} + \nabla\times\left(\vec{v}\times\vec{B}\right)$$

Induction equation

Magnetohydrodynamics

MHD approximations

- **Based on characteristics numbers compare terms:**
  - Induction equations: $$\frac{\partial \vec{B}}{\partial t} = \nabla \times \left(\vec{v} \times \vec{B}\right) + \frac{1}{\mu\sigma}\Delta\vec{B}$$
- Same dimensional estimates as for HD Navier-Stokes equation
  - assume stationary flow: $\frac{\partial \vec{B}}{\partial t} = 0$
  - Estimate through gradients over characteristic scale: L
  - Compare source (diffusion) term and curl motion: $\frac{\vec{B}}{\mu\sigma L^2} \leftrightarrow \frac{vB}{L}$
  - Characteristic number: $R_m = vL\mu\sigma$ Magnetic Reynolds number
  - If $R_m \gg 1$ magnetic dissipation is not relevant $\rightarrow$ perfect conductivity

$R_m \gg 1$

- Example for cold atomic ISM: $\mu\sigma = 10^{-22}\,\frac{s}{cm^2} \rightarrow R_m = 300$

Ideal MHD

- **Stronger approximation** $\mathcal{R}m \gg 1$ **= high conductivity** $\sigma \to \infty$
  - Reduction of $\vec{j} = \sigma(\vec{E} + \vec{v} \times \vec{B})$ $\to$ $\vec{E} = -\vec{v} \times \vec{B}$
    $\nabla \times \vec{E} = -\dot{\vec{B}}$ $\to$ $\nabla \times (\vec{v} \times \vec{B}) = \dot{\vec{B}}$

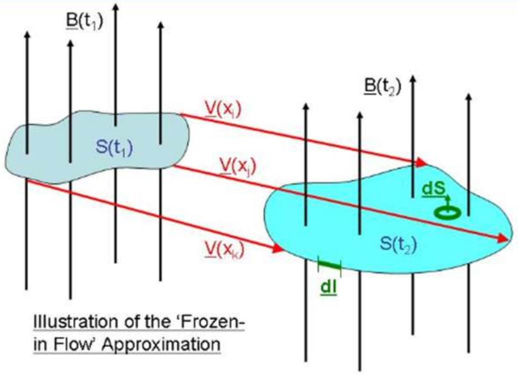
    (induction equation of ideal MHD)
  - $\vec{E}, \vec{H}$ drop out
  - Together with $\nabla \vec{B} = 0$ only 2 equations for electromagnetic interactions
  - Current $\vec{j}$ can be directly computed, not evolving independently

    $$\nabla \times \vec{H} = \vec{j}, \qquad \vec{B} = \mu \vec{H} \to \vec{j} = \frac{1}{\mu} \nabla \times \vec{B}$$
  - Most codes nowadays assume ideal MHD
  - Works well for ionization degrees as low as 10^{-5}.
  - But ignores frictional energy loss between neutrals and ions/electrons

Consequence

Frozen flux theorem – Alfvén‘s theorem

- Consider magnetic flux through any co-moving surface $\vec{S}$ in the fluid



$$\frac{d\Phi}{dt} = \iint \frac{\partial \vec{B}}{\partial t} d\vec{S} + \oint \vec{B}\,\vec{v} \times d\vec{l}$$

change in B   change in S

- Replace $\partial B/\partial t$ using induction equation and vector relation

$$\left(\vec{a} \times \vec{b}\right) \cdot \vec{c} = -(\vec{a} \times \vec{c}) \cdot \vec{b}$$

$$\frac{d\Phi}{dt} = \iint \nabla \times (\vec{v} \times \vec{B})\, d\vec{S} - \oint (\vec{v} \times \vec{B})\, d\vec{l}$$

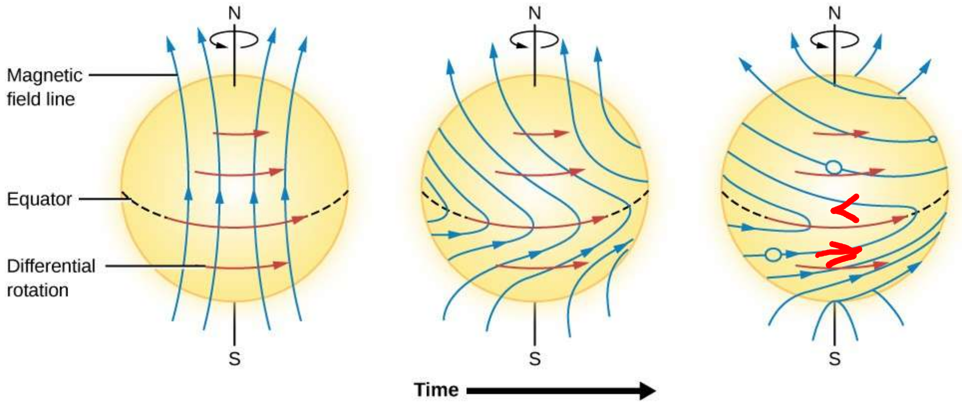
- Use Stokes theorem

$$\iint \left(\nabla \times \vec{F}\right) d\vec{S} = \oint \vec{F} d\vec{l} \quad \rightarrow \quad \frac{d\Phi}{dt} = 0$$

Consequence

Frozen flux theorem – Alfvén's theorem

- $\frac{d\Phi}{dt} = 0$
- The magnetic flux passing through any surface $\vec{S}$ moving along with the fluid is conserved
- Impact on the flow:
  - Free flow of ionized material along the field lines
  - For any perpendicular motion, the field lines will push the fluid or they will be dragged with the fluid.
  - Compression of the gas must include compression of the field lines $\rightarrow$ increases magnetic pressure.



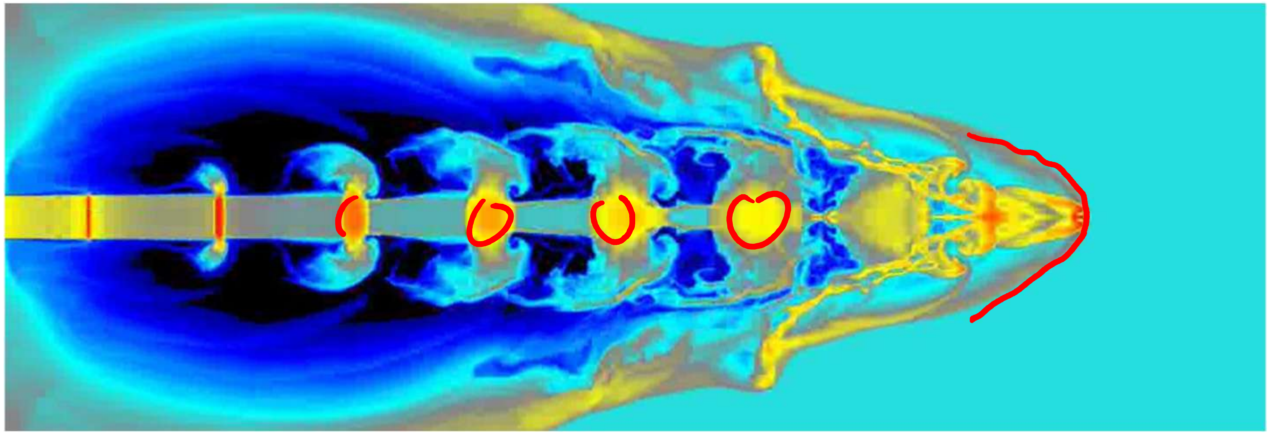
Magnetohydrodynamics

Overall system

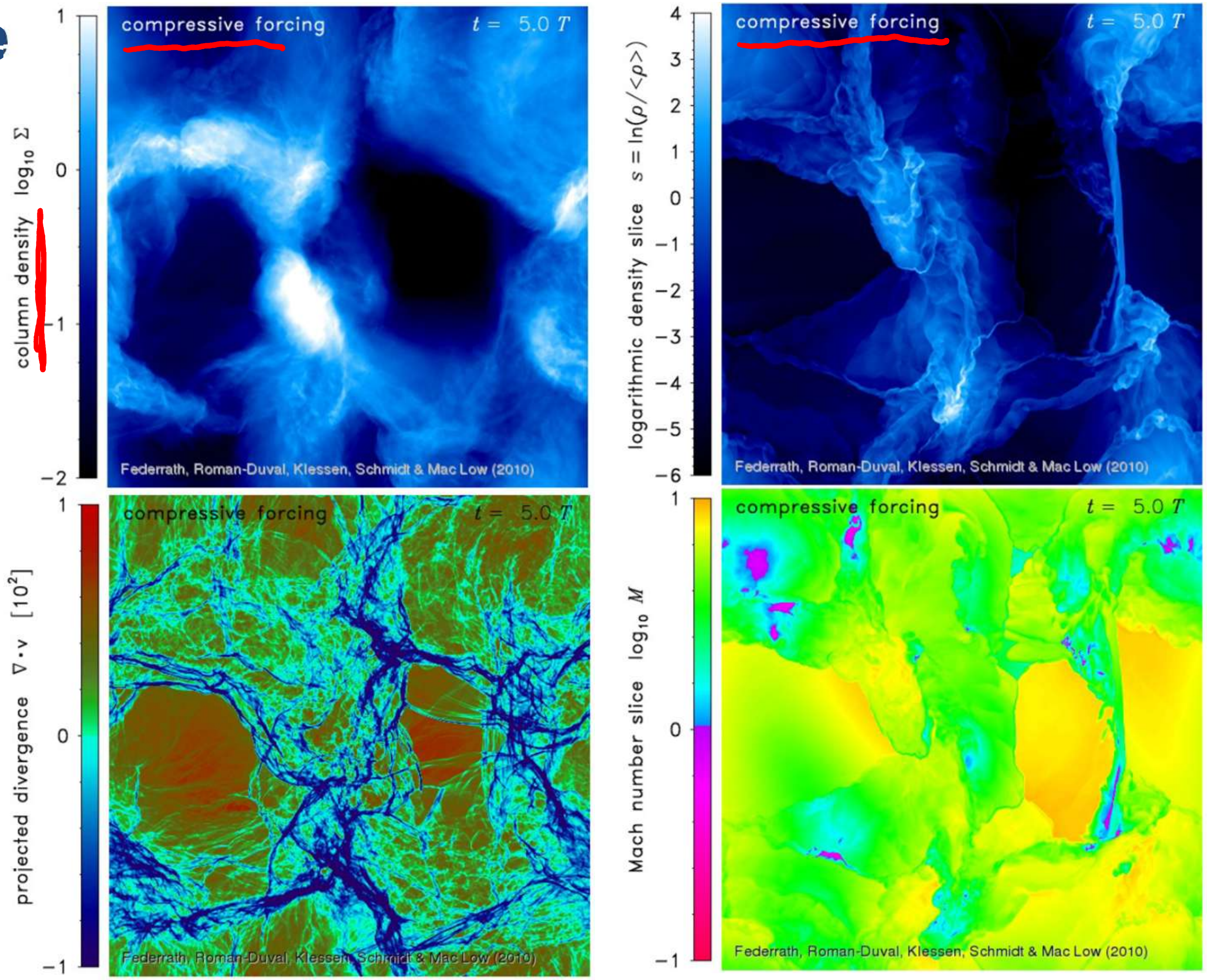
- **6 HD equations** (reduced to 4 in case of polytropic gas)
- **4 MHD electrodynamic equations**
  - to be solved numerically
    - **temporal evolution of interstellar gas**

- **Applications: Expansion of a magnetized jet**

Stone & Norman (1993)
(jetmovie.mpg)



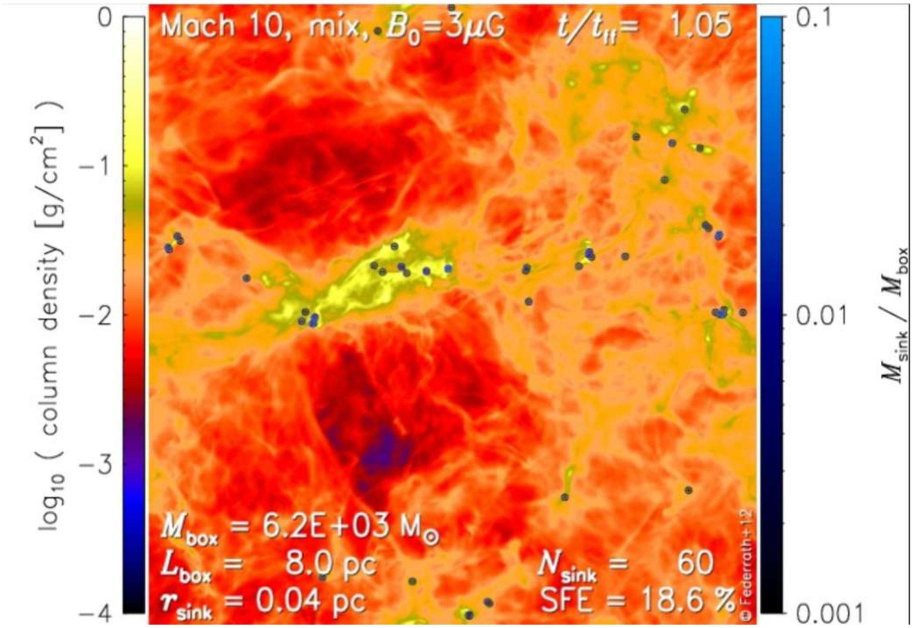
Dynamics in the diffuse interstellar medium



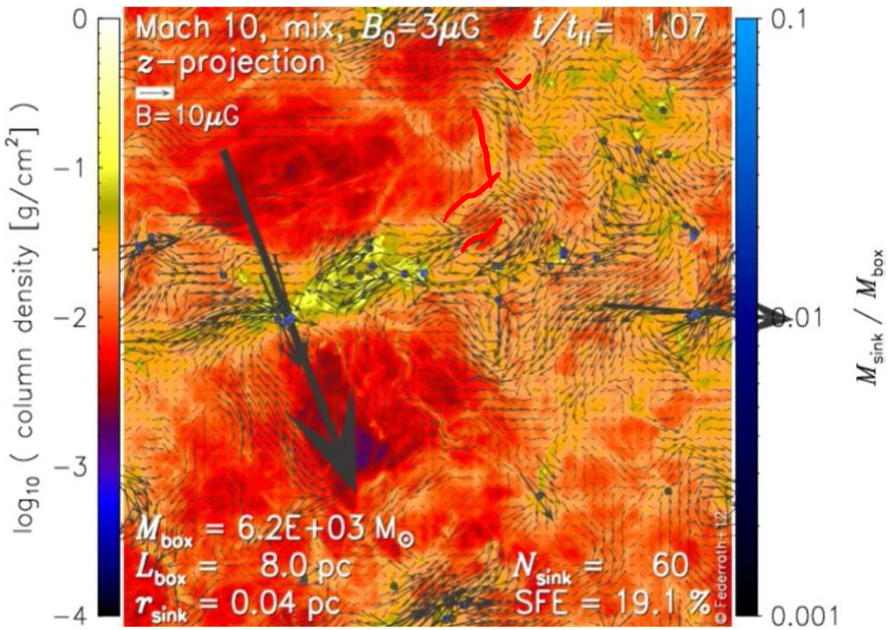
Federrath et al. (2010)

MHD Simulations

With gravitational collapse



Federrath_SFR_mM10B3.mp4
Federrath_SFR_mM10B3vec.mp4



Federrath et al. (2010)

The dynamics of interstellar gas

2. Instabilities and Turbulence

2.1 Jeans instability

2.2 Rayleigh-Taylor

2.3 Kelvin-Helmholtz

2.4 Parker instability

2.5 Turbulence

Kolmogorov scaling

Observations

Scaling relations

Gravitational instability – Jeans instability

- Pure hydrodynamics, i.e. neglect of $\vec{j}$ term
- Perturbation analysis for „smooth“ medium:
  - Assume
    - Extended (inifinite) medium
    - Constant density ρ_0
    - At rest $\vec{v}_0 = 0$
  - Evaluate
    - Evolution of a small density perturbation $\rho = \rho_0 + \rho_1, \quad \rho_1 \ll \rho_0$

$$\frac{d\rho}{dt} = \frac{d\rho_1}{dt} \qquad \vec{v} = \vec{v}_1$$

    - Small perturbation does not lead to large velocity gradients $\rightarrow \frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t}$ and friction terms can be neglected

Gravitational Instability

Use simplified hydrodynamic equations (1-4)

1. **Navier-Stokes:** $\rho \frac{d\vec{v}}{dt} = -\rho \nabla U_{grav} - \nabla p$

$\frac{\partial \vec{v}_1}{\partial t} = -\nabla U_{gr.} - \frac{1}{\rho_0} \nabla p$

-only perturbation gives pressure gradient
-no large scale flow: $d\vec{v}/dt = \partial\vec{v}/\partial t$
-only ρ_0 determines scaling factor, $\rho \approx \rho_0$

2. **Poisson:** $\Delta U_{grav} = 4\pi G\rho$

$\Delta U_{gr.} = 4\pi G \rho_1$

-only density perturb. contributes to U_{grav}
-infinitely extended medium gives constant shift

3. **Continuity:** $\frac{\partial\rho}{\partial t} = \nabla(\rho\vec{v}) = 0$

$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \vec{v}_1 = 0$

-only ρ_0 determines scaling factor

Use simplified hydrodynamic equations (1-4)

4. Equation of state:

- Use polytropic equation of state
  - Independent of *T*
  - Only 4 equations needed

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma} \quad \text{or} \quad c_s^2 = \frac{dp}{d\rho} = \gamma \frac{\mathrm{p}}{\rho}$$

- $1 + \frac{p_1}{p_0} = \left(1 + \frac{\rho_1}{\rho_0}\right)^{\gamma}$ or $c_s^2 = \gamma \frac{p_0}{\rho_0}$

- **Use with pressure term in Navier-Stokes:**

$$\nabla p_1 = \frac{dp}{d\rho} \nabla \rho_1 = c_s^2 \nabla \rho_1$$

$*$