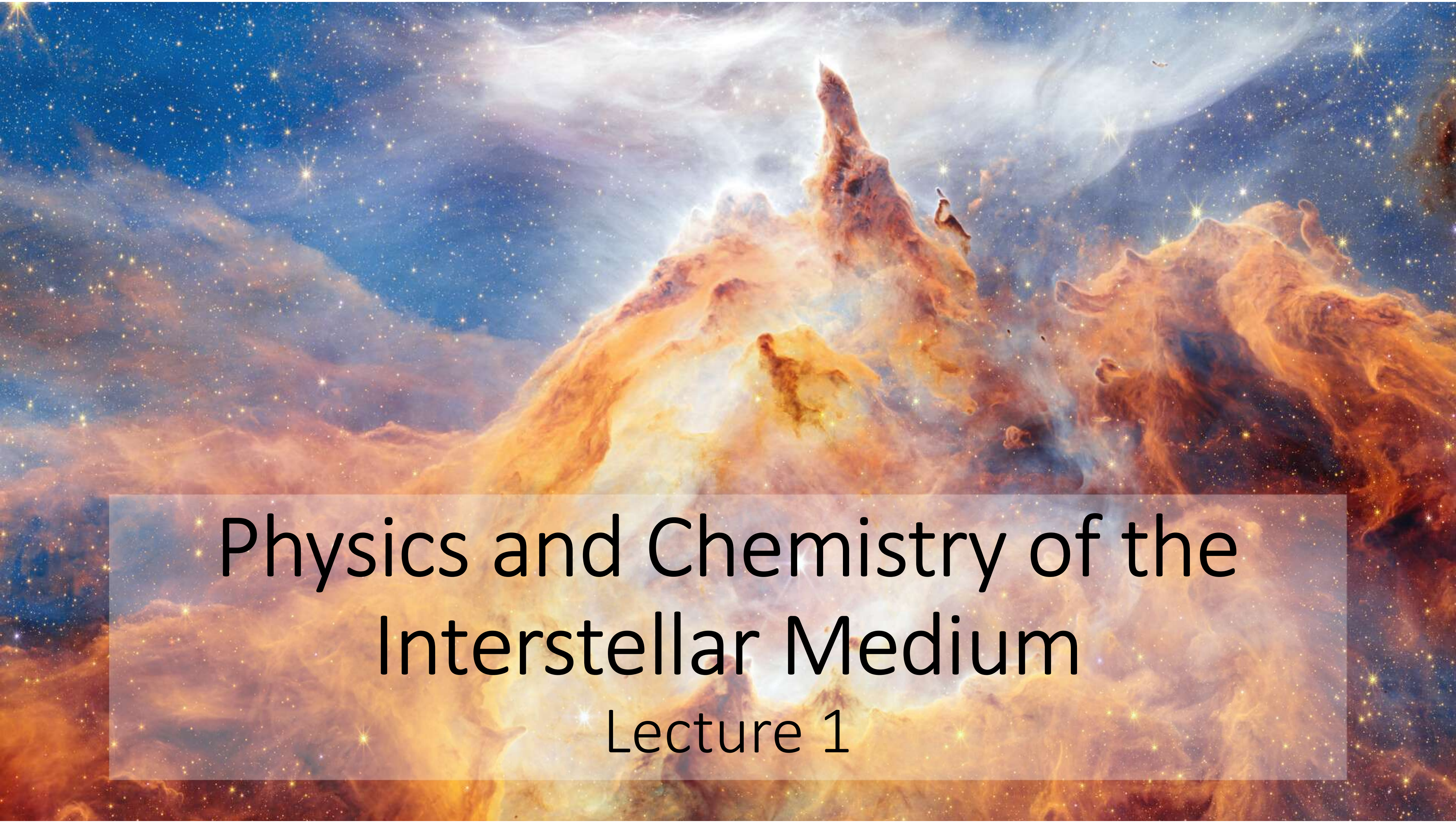


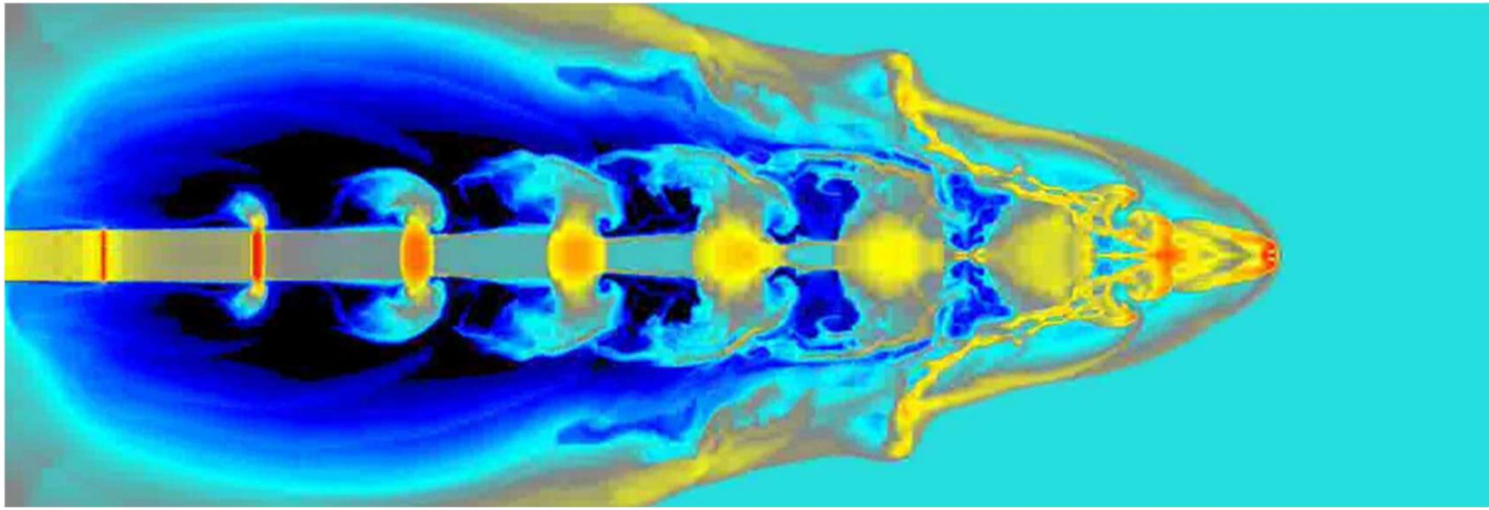
Physics and Chemistry of the Interstellar Medium

Lecture 1

The dynamics of interstellar gas

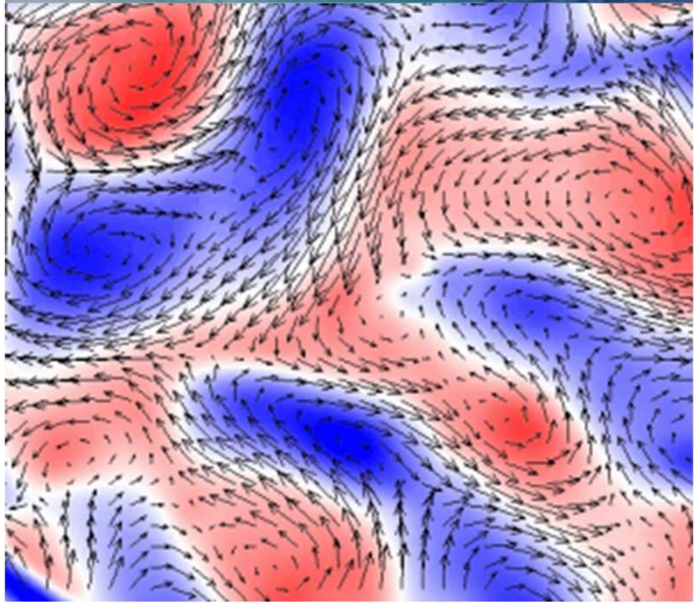
Lecture I

1. Hydrodynamics and Magnetohydrodynamics
    1.1. Fundamental equations



Why Hydrodynamics?

- Motion of gas in space
    - Compressible Hydrodynamics
- Follow the cells of gas in the fluid
    - Intrinsic properties
        - (Mass) density ρ
        - Cell velocity $\vec{v}$
        - Viscosity η
        - Conductivity σ
        - ...
    - Normalize all quantities to volume
        - Force density $\vec{k}$
        - (Electric) current density $\vec{j}$
        - …



The hydrodynamic equations

- **Equation of motion (Navier-Stokes-Equation)**

$$\rho \frac{d\vec{v}}{dt} = \vec{k}$$

$\vec{k}$ = force density
ρ = mass density

material

  - Total derivative:

$$\frac{D\vec{v}}{Dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}$$

acceleration + advection (conv. acceleration)

  - Example: **spherical geometry:**

$$\frac{D\vec{v}}{Dt} = \frac{dv}{dt} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r}$$

- **Forces**:

$$\vec{k} = \vec{k}_{mag} + \vec{k}_{grav} + \vec{k}_{press} + \vec{k}_{frict}$$

Fundamental equations

The hydrodynamic equations

- **Forces**

$$\vec{k} = \vec{k}_{mag} + \vec{k}_{grav} + \vec{k}_{press} + \vec{k}_{frict}$$

  - Magnetic fields: $\vec{k}_{mag} = \vec{j} \times \vec{B}$ — $\vec{j}$: el. current density
  - Gravitation: $\vec{k}_{grav} = -\rho \nabla U_{grav}$ — U_{grav} : grav. Potential
  - Pressure: $\vec{k}_{press} = -\nabla p$ — p : pressure

Fundamental equations

The hydrodynamic equations

- **Forces**

$$\vec{k} = \vec{k}_{mag} + \vec{k}_{grav} + \vec{k}_{press} + \vec{k}_{frict}$$

  - Friction: $$\vec{k}_{frict} = \eta \Delta \vec{v} + \left(\frac{\eta}{3+\zeta}\right) \nabla(\nabla \vec{v})$$ for isotropic medium

η : dynamical viscosity Scherkräfte

ζ : volume viscosity (both assumed to be constant)

Fundamental equations

- Gradient $grad\ f = \nabla f \qquad \nabla = \sum_{i=1}^{n} \vec{e}_i \frac{\partial}{\partial x_i} = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_i}\right)$
- Divergence $div\ f = \nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$
- Curl $curl\ f = \nabla \times \vec{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\vec{e}_x + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\vec{e}_y + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\vec{e}_z$
- Laplacian $\Delta\vec{v} = \nabla \cdot \nabla = \nabla^2$

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} \qquad \Delta\vec{v} = \sum_{i=1}^{n} \frac{\partial^2 v_i}{\partial x_i^2}\vec{e}_i$$

Physical interpretation $\eta\Delta\vec{v}$

example in 1-dim: $\Delta v = \frac{\partial^2 v}{\partial x^2}$ (minimum: $\Delta v > 0$, maximum $\Delta v < 0$)

$\eta\Delta v > 0$ => acceleration due to friction

$\eta\Delta v < 0$ => decceleration due to friction

Physical interpretation $\left(\frac{\eta}{3+\zeta}\right)\nabla(\nabla\vec{v})$

- $\nabla\vec{v} > 0$ expansion (fluid volume increases)
- $\nabla\vec{v} < 0$ compression (fluid volume decreases)
- $\nabla\vec{v} = 0$ incompressible (fluid volume is conserved)

- How is $\nabla\vec{v}$ a rate of volume change?
  - Fluid element with volume $V = dx\,dy\,dz$
  - Velocity field $\vec{v} = \left(v_x, v_y, v_z\right)$

The equation of motion

- **Underlying assumptions:**
  - ‚Material constants' (fluid properties):

η	=	dynamical viscosity
ζ	=	volume viscosity
σ	=	conductivity
X	=	degree of ionization

    - Do not change with fluid motion or on small scales as function of $\vec{r}$

  - Concept of fluid mechanics: **follow the gas**

Lagrange

Fundamental equations

The hydrodynamic equations

- **The continuity equation** **(no mass production)**

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \vec{v}) = 0$$

  - Example: **spherical geometry**

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho\, v(r)\right) = 0$$

- **Poisson equation** **(Gravitational potential)**

$$\Delta U_{grav} = 4\pi G \rho$$

G : Gravitational constant

  - Example: **spherical geometry**

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial U_{grav}}{\partial r^2}\right) = 4\pi G \rho(r)$$

Fundamental equations

The hydrodynamic equations

- **The equation of state (EOS)**
  - Relates density ρ and pressure p
  - Closes system for quantities $\underline{\rho, \vec{v}, p, U_{grav}}$
  - Derive from Thermodynamics:
    - 1. Law of TD $\quad dQ = dE + p\,dV$
    - Heat capacity $\quad c = \frac{dQ}{dT} = \frac{dE}{dT} + p\frac{dV}{dT}$
      - At constant volume: $\quad c_V = \frac{dE}{dT}$
    - Gibbs enthalpy $\quad dH = d(E + pV)$
      - Combine: $\quad dQ = dH - V\,dp$
        - Heat capacity at constant pressure $\quad c_p = \frac{dH}{dT}$

The equation of state

Thermodynamics

- Consider **adiabatic** process (Isentrope) $S = const.$
  - Heat exchange: $$dQ = T dS = 0$$
  - For inner energy: $$\frac{dU}{dT} = c_V = -p\frac{dV}{dT}$$
  - For Gibbs enthalpy: $$\frac{dH}{dT} = c_p = V\frac{dp}{dT}$$ $$\frac{dH}{dT} = c_p = V \cdot \frac{dp}{dT}$$
    - Gas coefficients c_p, c_V are constants

$$\rightarrow \frac{c_p}{c_V} =: \gamma = -\frac{V}{p}\frac{dp}{dV}$$

  - Separation of variables: $$\frac{dp}{p} = -\gamma\frac{dV}{V}$$

The equation of state

Thermodynamics

- Consider **adiabatic** process
  - Separation of variables: $\frac{dp}{p} = -\gamma \frac{dV}{V}$
  - Solution: $\ln p = -\gamma \ln V + \text{const.}$ $\quad p \propto V^{-\gamma}$
  - Translation into density: $\rho \propto \frac{1}{V}$

    Polytropic equation of state: $p \propto \rho^{\gamma}$
- Alternative notation:
  - Sound speed instead of polytropic coefficient: $c_s^2 = \frac{dp}{d\rho} = \gamma \frac{p}{\rho}$
  - Differential expression

    (sound speed = expansion velocity of a density perturbation $\rho = \rho_0 + \delta\rho$)

Fundamental equations

The hydrodynamic equations

- **Navier-Stokes + Poisson + continuity + EOS**
  = closed system of 4 hydrodynamic equations

- **But**: Polytropic equation of state = over-simplification
  - Energy exchange with environment
  - Inner energy may change
    - Friction
    - Electric heat dissipation
    - Chemical transition, radiation
  - General equation for $e(\rho, T)$ needed
    - Notation: E = inner energy of given volume e = inner energy density

The hydrodynamic equations

- The caloric equation of state $e(\rho, T)$
  - Very general, in principal including many effects
  - Here only **ideal gas**

$$e = c_V T = \frac{n_f}{2} \frac{k_B}{\mu_{mol}} T$$

k_B Boltzmann constant

μ_{mol} molecular mass of gas particle

n_f number of internal degrees of freedom (DOF) per gas particle (includes rotational degrees of freedom)

3 – for monoatomic gas (He)
5 – for linear molecules (H_2,CO) ($+3N-5$ if vibrations included)
6 – for non-linear molecules (CH_3OH) ($+3N-6$ vibrational DOFs for $N > 2$)

The caloric equation of state

Ideal gas

- Express γ through n_f

$$\gamma = \frac{c_p}{c_V} \qquad c_p = \frac{d\left(e + \frac{P}{\rho}\right)}{dT} \qquad c_V = \frac{n_f}{2}\frac{k_B}{\mu_{mol}}$$

$$c_p = c_V + \frac{d\left(\frac{P}{\rho}\right)}{dT}$$

- Ideal gas: $\frac{P}{\rho} = \frac{k_B T}{\mu_{mol}}$
- Combine: $c_p = \frac{n_f}{2}\frac{k_B}{\mu_{mol}} + \frac{k_B}{\mu_{mol}} = \left(1 + \frac{2}{n_f}\right)\frac{n_f}{2}\frac{k_B}{\mu_{mol}}$
- Polytropic oefficient:

$$\gamma = 1 + \frac{2}{n_f}$$

The caloric equation of state

Ideal gas

- Polytropic coefficient γ: $\gamma = 1 + \frac{2}{n_f}$

  $5/3$ – for monoatomic gas (He)
  $7/5$ – for two-atomic gas (H_2,CO)
  $4/3$ – for multi-atomic gas

- Polytropic equation of state: $p \propto \rho^{\gamma}$
  - Ideal gas: $p = \frac{k_B}{\mu_{mol}} \rho T$
  - Combines to: $\rho^{\gamma-1} \sim \frac{k_B}{\mu_{mol}} T \rightarrow e \sim \rho^{\gamma-1}$
  - Isothermal processes: $\gamma = 1$
    (when inner energy is dominated by **radiation equilibrium** with surrounding)

The caloric equation of state

Ideal gas

- Use equation of state: $p = \frac{k_B}{\mu_{mol}} \rho \mathrm{T}$
- To compute sound speed: $c_s^2 = \frac{\partial p}{\partial \rho} = \gamma \frac{p}{\rho}$ $\rightarrow c_s = \sqrt{\frac{\gamma k_B T}{\mu_{mol}}}$
  - Examples:
    - Earth atmosphere: N_2: $\gamma = \frac{7}{5}, \mu_{mol} = 4.7 \times 10^{-26} kg, T = 293\ K$
      $\rightarrow c_s = 350\ m/s$
    - Atomic ISM: HI +10% He: $\gamma = \frac{5}{3}, \mu_{mol} = 2.2 \times 10^{-27} kg, T = 100\ K$
      $\rightarrow c_s = 1\ km/s$
  - Notation: Use particle number density: $n = \frac{\rho}{\mu_{mol}} \rightarrow p = n\, k_B T$ $[cm^{-3}]$

But interstellar gas is not adiabatic! $\rightarrow$ **Heat sinks and sources** *